Lecture notes on p-adic Hodge theory

Serin Hong

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CHAPTER I

Introduction

1. A first glimpse of p-adic Hodge theory

Our goal in this section is to give a brief overview of p-adic Hodge theory. By nature, p-adic Hodge theory admits two different perspectives, namely the arithmetic one and the geometric one. We illustrate some key ideas of p-adic Hodge theory from each perspective, and discuss how the two perspectives are related.

1.1. The arithmetic perspective

A central object in algebraic number theory is the absolute Galois group $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Indeed, $\Gamma_{\mathbb{Q}}$ contains virtually all arithmetic information about the field $\mathbb{Q}$ (and its finite extensions, called *number fields*). However, since $\Gamma_{\mathbb{Q}}$ is an extremely sophisticated object, we usually study it via the natural injective group homomorphism $\Gamma_{\mathbb{Q}_p} \hookrightarrow \Gamma_{\mathbb{Q}}$ induced by the canonical embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ for each prime p. It is a general principle that we can deduce much information about $\Gamma_{\mathbb{Q}}$ from knowledge about $\Gamma_{\mathbb{Q}_p}$ for each prime p.

The group $\Gamma_{\mathbb{Q}_p}$ is still quite complicated, but turns out to be much more manageable than the group $\Gamma_{\mathbb{Q}}$ is. The main objective of p-adic Hodge theory, from the arithmetic perspective, is to understand $\Gamma_{\mathbb{Q}_p}$ via continuous representations $\Gamma_{\mathbb{Q}_p} \to \mathrm{GL}_n(\mathbb{Q}_p)$, called *$p$-adic Galois representations*, where $\Gamma_{\mathbb{Q}_p}$ and $\mathrm{GL}_n(\mathbb{Q}_p)$ are respectively endowed with the profinite topology and the p-adic topology. Such representations are particularly interesting as they encode two different kinds of structures on $\mathbb{Q}_p$, namely the algebraic ones from the group $\Gamma_{\mathbb{Q}_p}$ and the analytic ones from the p-adic topology.

In this subsection, we present a primary example that shows why p-adic Galois representations are important for carrying out the strategy outlined in the first paragraph and how we study such representations. Let E be an *elliptic curve* over $\mathbb{Q}$, which refers to a projective curve defined by a polynomial equation

$$y^2 = x^3 + ax + b \quad \text{with } a, b \in \mathbb{Q} \text{ and } 4a^3 + 27b^2 \neq 0. \tag{1.1}$$

Elliptic curves play a fundamental role in modern number theory, as highlighted by the proof of Fermat's last theorem. Elliptic curves have a remarkable property that their points (including the point at infinity) naturally form an abelian group. Hence for each positive integer n and a $\mathbb{Q}$-algebra R, we can define

$$E[n](R) := \{P \in E(R) : nP = O\}$$

where O denotes the point at infinity identified as the zero element in E. We fix a prime ℓ and define the *ℓ-adic Tate module* of E by

$$T_\ell(E) := \varprojlim E[\ell^v](\overline{\mathbb{Q}})$$

where the transition maps send each $P \in E[\ell^{v+1}](\overline{\mathbb{Q}})$ to $\ell P \in E[\ell^v](\overline{\mathbb{Q}})$. It is a standard fact that $T_\ell(E)$ is a free $\mathbb{Z}_\ell$-module of rank 2, thereby admitting an isomorphism

$$T_\ell(E) \simeq \mathbb{Z}_\ell^2.$$

Moreover, the tautological action of $\Gamma_{\mathbb{Q}}$ on $\overline{\mathbb{Q}}$ naturally induces a continuous action on $T_\ell(E)$, and in turn gives rise to a continuous representation of $\Gamma_{\mathbb{Q}}$ on

$$V_\ell(E) := T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell^2$$

called the *ℓ-adic rational Tate module* of E. The action of $\Gamma_{\mathbb{Q}}$ on $T_\ell(E)$ and $V_\ell(E)$ contains much information about the elliptic curve E, as suggested by the following fact:

THEOREM 1.1.1 (Faltings [**Fal83**]). Given two elliptic curves E_1 and E_2 over $\mathbb{Q}$, there exist natural isomorphisms

$$\begin{aligned} \mathrm{Hom}(E_1, E_2) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell &\cong \mathrm{Hom}_{\Gamma_{\mathbb{Q}}}(T_\ell(E_1), T_\ell(E_2)), \\ \mathrm{Hom}(E_1, E_2) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell &\cong \mathrm{Hom}_{\Gamma_{\mathbb{Q}}}(V_\ell(E_1), V_\ell(E_2)). \end{aligned} \tag{1.2}$$

In particular, a homomorphism between E_1 and E_2 is uniquely determined by the induced map on the Tate modules as $\Gamma_{\mathbb{Q}}$-representations.

Remark. By a result of Tate [**Tat66**], an analogous statement holds for elliptic curves over $\mathbb{F}_p$ with $p \neq \ell$. Both Theorem 1.1.1 and the result of Tate [**Tat66**] are special cases of the *Tate conjecture* which relates subvarieties of a given algebraic variety X over a field k to representations of $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$ on vector spaces over $\mathbb{Q}_\ell$ that naturally arise from X (similar to the ℓ-adic rational Tate module an elliptic curve). For elliptic curves over $\mathbb{Q}_p$, we get injective maps instead of isomorphisms in (1.2).

However, the action of $\Gamma_{\mathbb{Q}}$ on $T_\ell(E)$ and $V_\ell(E)$ is difficult to understand due to the complexity of the group $\Gamma_{\mathbb{Q}}$. Following the strategy outlined at the beginning of this subsection, we study the action of $\Gamma_{\mathbb{Q}_p}$ on $T_\ell(E)$ and $V_\ell(E)$ for each prime p via the natural injection $\Gamma_{\mathbb{Q}_p} \hookrightarrow \Gamma_{\mathbb{Q}}$. In fact, we have an identification

$$T_\ell(E) \cong \varprojlim E[\ell^v](\overline{\mathbb{Q}}_p) \simeq \mathbb{Z}_\ell^2,$$

endowed with a continuous action of $\Gamma_{\mathbb{Q}_p}$ naturally induced by the tautological action on $\overline{\mathbb{Q}}_p$.

We assume that E has good reduction at p. For $p > 3$, our assumption concretely means that in the polynomial equation (1.1) we have $a, b \in \mathbb{Z}_p$ with $4a^3 + 27b^2$ not divisible by p. The assumption is not very restrictive; indeed, it is a standard fact that E has good reduction at almost all primes (i.e., all but finitely many primes). A main consequence of our assumption is that E admits mod p reduction, denoted by $\overline{E}$, which is an elliptic curve over $\mathbb{F}_p$ with points given by the mod p solutions of (1.1). We have the ℓ-adic Tate module of $\overline{E}$ defined by

$$T_\ell(\overline{E}) := \varprojlim \overline{E}[\ell^v](\overline{\mathbb{F}}_p),$$

which turns out to be a free module over $\mathbb{Z}_\ell$ (but not necessarily of rank 2) with a continuous action of $\Gamma_{\mathbb{F}_p} = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ naturally induced by the tautological action on $\overline{\mathbb{F}}_p$, and consequently obtain a continuous representation of $\Gamma_{\mathbb{F}_p}$ on the ℓ-adic rational Tate module

$$V_\ell(\overline{E}) := T_\ell(\overline{E}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

For $p \neq \ell$, we can explicitly describe the action of $\Gamma_{\mathbb{Q}_p}$ on $T_\ell(E)$ and $V_\ell(E)$ through the action of $\Gamma_{\mathbb{F}_p}$ on $T_\ell(\overline{E})$ and $V_\ell(\overline{E})$. In fact, if we regard $T_\ell(\overline{E})$ and $V_\ell(\overline{E})$ as $\Gamma_{\mathbb{Q}_p}$-representations via the natural surjection $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p) \cong \Gamma_{\mathbb{F}_p}$, where $\mathbb{Q}_p^{\mathrm{un}}$ denotes the maximal unramified extension of $\mathbb{Q}_p$, we have isomorphisms

$$T_\ell(E) \simeq T_\ell(\overline{E}) \qquad \text{and} \qquad V_\ell(E) \simeq V_\ell(\overline{E})$$

as $\Gamma_{\mathbb{Q}_p}$-representations. Hence we only need to understand $T_\ell(\overline{E})$ and $V_\ell(\overline{E})$ as (continuous) $\Gamma_{\mathbb{F}_p}$-representations. The group $\Gamma_{\mathbb{F}_p}$ is topologically generated by the Frobenius automorphism which maps each element in $\overline{\mathbb{F}}_p$ to its p-th power. It turns out that the Frobenius automorphism acts on $T_\ell(\overline{E})$ and $V_\ell(\overline{E})$ with characteristic polynomial $x^2 - a_p x + p$, where we set $a_p := p + 1 - \#\overline{E}(\mathbb{F}_p)$. In summary, we can specify the action of $\Gamma_{\mathbb{Q}_p}$ on $T_\ell(E)$ and $V_\ell(E)$ by the following properties:

(i) The action is continuous and factors through the natural surjection $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p}$.

(ii) The Frobenius automorphism of $\overline{\mathbb{F}}_p$, which topologically generates $\Gamma_{\mathbb{F}_p}$, acts with trace $a_p = p + 1 - \#\overline{E}(\mathbb{F}_p)$ and determinant p.

We refer to a $\Gamma_{\mathbb{Q}_p}$-representation with property (i) as an *unramified* representation, motivated by the natural identification $\Gamma_{\mathbb{F}_p} \cong \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$. Since the ℓ-adic Tate module $T_\ell(E)$ is unramified, it loses much information about the topology on $\Gamma_{\mathbb{Q}_p}$; indeed, the topology on $\Gamma_{\mathbb{F}_p}$ is very simple (being generated by one element, namely the Frobenius automorphism) compared to the topology on $\Gamma_{\mathbb{Q}_p}$. Intuitively, for $p \neq \ell$ the topologies on $\Gamma_{\mathbb{Q}_p}$ and $\mathbb{Q}_\ell$ do not get along with each other very well, and in turn force the continuous action of $\Gamma_{\mathbb{Q}_p}$ on $T_\ell(E)$ to be simple. It is worthwhile to mention that our discussion here explains one direction of the following important criterion:

Theorem 1.1.2 (Néron [**Nér64**], Ogg [**Ogg67**], Shafarevich). An elliptic curve E over $\mathbb{Q}$ has good reduction at $p \neq \ell$ if and only if $T_\ell(E)$ is unramified.

Let us now set $p = \ell$. We have entered the realm of p-adic Hodge theory, as $V_p(E)$ is a p-adic Galois representation by construction. In stark contrast to our discussion in the previous two paragraphs, we have the following facts:

(1) The (rational) Tate modules for E and $\overline{E}$ are never isomorphic; indeed, $T_p(\overline{E})$ is isomorphic to either $\mathbb{Z}_p$ or 0 whereas $T_p(E)$ is always isomorphic to $\mathbb{Z}_p^2$.

(2) $T_p(E)$ and $V_p(E)$ turn out to be never unramified; in other words, the action of $\Gamma_{\mathbb{Q}_p}$ on $T_p(E)$ and $V_p(E)$ always has a nontrivial contribution from the kernel of the surjection $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p}$, called the *inertia group* of $\mathbb{Q}_p$ and denoted by $I_{\mathbb{Q}_p}$.

The second fact indicates that the topologies on $\Gamma_{\mathbb{Q}_p}$ and $\mathbb{Q}_p$ do not clash and thus allow $T_p(E)$ to carry a large amount of topological information. A side effect is that, as the first fact shows, it is impossible to describe $T_p(E)$ solely based on $T_p(\overline{E})$.

We still wish to understand $T_p(E)$ as a $\Gamma_{\mathbb{Q}_p}$-representation using the mod p reduction $\overline{E}$. Following Tate [**Tat66**] and Grothendieck [**Gro71, Gro74**], we regard E as a curve over $\mathbb{Z}_p$ and consider the functors defined by

$$E[p^\infty] := \varinjlim E[p^v] \quad \text{and} \quad \overline{E}[p^\infty] := \varinjlim \overline{E}[p^v],$$

called the *p-divisible groups* of E and $\overline{E}$, where the transition maps are the natural inclusions. For the elliptic curve E, the p-divisible group $E[p^\infty]$ and the Tate module $T_p(E)$ are equivalent objects in the sense that we can determine one from the other. On the other hand, for the mod p reduction $\overline{E}$, the p-divisible group $\overline{E}[p^\infty]$ contains a lot of information that the Tate module $T_p(\overline{E})$ does not; for example, $\overline{E}[p^\infty]$ never vanishes while $T_p(\overline{E})$ often does (as noted in the previous paragraph). Hence the p-divisible groups serve as refinements of the p-adic Tate modules which do not lose too much information under mod p reduction.

A remarkable fact is that we can describe p-divisible groups in terms of linear algebraic objects. A *Dieudonné module* over $\mathbb{Z}_p$ refers to a finite free $\mathbb{Z}_p$-module M equipped with an endomorphism φ_M, called the *Frobenius endomorphism*, such that $\varphi_M(M)$ contains pM. A *Honda system* over $\mathbb{Z}_p$ is a Dieudonné module M over $\mathbb{Z}_p$ together with a submodule $\mathrm{Fil}^1(M)$ such that φ_M induces a natural isomorphism $\mathrm{Fil}^1(M)/p\,\mathrm{Fil}^1(M) \cong M/\varphi_M(M)$.

THEOREM 1.1.3 (Dieudonné [**Die55**], Fontaine [**Fon77**]). Given an elliptic curve E over $\mathbb{Q}$ with good reduction at p, we have the following statements:

(1) The mod p reduction $\overline{E}$ of E functorially gives rise to a Dieudonné module $\mathbb{D}(\overline{E})$ over $\mathbb{Z}_p$ of rank 2, which uniquely determines the isomorphism class of $\overline{E}[p^\infty]$.
(2) For $p > 2$, the elliptic curve E functorially gives rise to a Honda system over $\mathbb{Z}_p$ with underlying Dieudonné module $\mathbb{D}(\overline{E})$, which uniquely determines the isomorphism class of $E[p^\infty]$.

Remark. Let us make some remarks regarding Theorem 1.1.3.

(1) The results of Dieudonné [**Die55**] and Fontaine [**Fon77**] indeed yield anti-equivalences of categories

$$\left\{\ p\text{-divisible groups over } \mathbb{F}_p\ \right\} \overset{\sim}{\longleftrightarrow} \left\{\ \text{Dieudonné modules over } \mathbb{Z}_p\ \right\}$$
$$\left\{\ p\text{-divisible groups over } \mathbb{Z}_p\ \right\} \overset{\sim}{\longleftrightarrow} \left\{\ \text{Honda systems over } \mathbb{Z}_p\ \right\}$$

where the second anti-equivalence holds only for $p > 2$. For $p = 2$, the second anti-equivalence holds after taking an appropriate subcategory on each side.
(2) The first statement, proved by Dieudonné [**Die55**], was the main motivation for Tate [**Tat66**] and Grothendieck [**Gro71, Gro74**] to study p-divisible groups in relation to the Tate modules, as it suggests that $\overline{E}[p^\infty]$ behaves much as $T_\ell(\overline{E})$ for $p \neq \ell$. The work of Tate [**Tat66**] and Grothendieck [**Gro71, Gro74**] eventually inspired the proof of the second statement by Fontaine [**Fon77**] in an attempt to describe $E[p^\infty]$ via $\mathbb{D}(\overline{E})$ together with some "lifting data".
(3) Our description of Dieudonné modules is potentially misleading. In general, for a Dieudonné module M the endomorphism φ_M should be Frobenius-semilinear in an appropriate sense. For Dieudonné modules over $\mathbb{Z}_p$, however, the Frobenius-semilinearity simply means linearity as the Frobenius automorphism is trivial on the residue field $\mathbb{F}_p$.

Hence for $p > 2$ we can determine the isomorphism class of $T_p(E)$ as a $\Gamma_{\mathbb{Q}_p}$-representation by the Honda system associated to E with underlying Dieudonné module $\mathbb{D}(\overline{E})$. Intuitively, once we fix an element $\sigma \in \Gamma_{\mathbb{Q}_p}$ that lifts the Frobenius automorphism in $\Gamma_{\mathbb{F}_p}$, the Honda system encodes the actions of $I_{\mathbb{Q}_p}$ and σ on $T_p(E)$ respectively by $\mathrm{Fil}^1(\mathbb{D}(\overline{E}))$ and $\varphi_{\mathbb{D}(\overline{E})}$. For $p = 2$, we can still associate a Honda system to E and show that it contains much information about $T_p(E)$, although in general it does not determine the isomorphism class of $T_p(E)$.

If we instead want to study the p-adic Galois representation on $V_p(E)$, we replace the Dieudonné module $\mathbb{D}(\overline{E})$ by $\mathbb{D}(\overline{E}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, called an *isocrystal* over $\mathbb{Q}_p$, which is a finite dimensional vector space over $\mathbb{Q}_p$ equipped with a (Frobenius-semilinear) automorphism. The Honda system associated to E yields the isocrystal $\mathbb{D}(\overline{E}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with the filtration given by the subspace $\mathrm{Fil}^1(\mathbb{D}(\overline{E})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, called a *filtered isocrystal* over $\mathbb{Q}_p$. Now Theorem 1.1.3 implies for $p > 2$ that the filtered isocrystal associated to E determines the isomorphism class of $V_p(E)$ as a p-adic Galois representation, which turns out to apply also for $p = 2$.

We have thus transferred the study of $T_p(E)$ and $V_p(E)$ as $\Gamma_{\mathbb{Q}_p}$-representations to the study of certain linear algebraic objects, such as Dieudonné modules and isocrystals. In fact, a main theme of p-adic Hodge theory is to construct a dictionary that relates p-adic Galois representations to various linear algebraic objects. Our discussion here illustrates a prototype for such a dictionary.

1.2. The geometric perspective

Our discussion in §1.1 shows how we can study elliptic curves over $\mathbb{Q}$ via their Tate modules as $\Gamma_{\mathbb{Q}}$-representations. It is natural to ask whether we can similarly study other algebraic varieties. Let X be a smooth proper variety over $\mathbb{Q}$. For each $\mathbb{Q}$-algebra R, we write X_R for the base change of X to R. Given an integer $n \geq 0$ and a prime ℓ, we have the *étale cohomology group* $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ which is a finite dimensional vector space over $\mathbb{Q}_\ell$ with a continuous action of $\Gamma_{\mathbb{Q}}$. As a special case, for an elliptic curve E over $\mathbb{Q}$ we have a natural identification

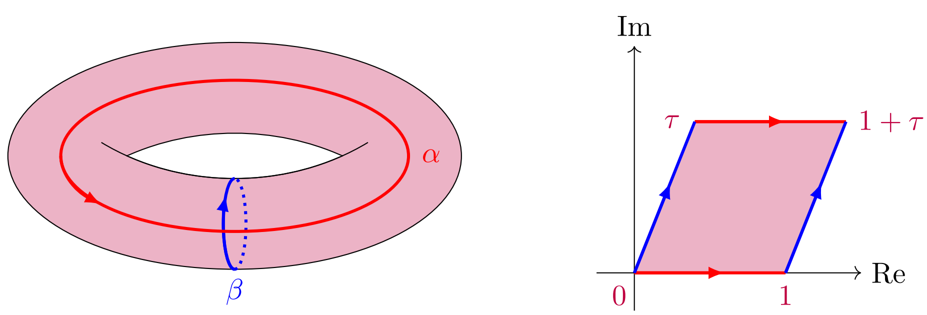
$$V_\ell(E)^\vee \cong H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$$

as $\Gamma_{\mathbb{Q}}$-representations, where $V_\ell(E)^\vee$ denotes the dual representation of $V_\ell(E)$. Following the strategy outlined in §1.1, for each prime p we study the action of $\Gamma_{\mathbb{Q}_p}$ on $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ via the natural injection $\Gamma_{\mathbb{Q}_p} \hookrightarrow \Gamma_{\mathbb{Q}}$; in other words, we study the étale cohomology group $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$ as a representation of $\Gamma_{\mathbb{Q}_p}$. For $p \neq \ell$, the $\Gamma_{\mathbb{Q}_p}$-representation $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$ tends to be simple; indeed, it is unramified for all but finitely many $p \neq \ell$, as we have already seen for the rational Tate modules of an elliptic curve in §1.1. For $p = \ell$, on the other hand, $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ as a p-adic Galois representation turns out to carry interesting information about the geometry of X. The main objective of p-adic Hodge theory, from the geometric perspective, is to extract information about the geometric structure of an algebraic variety from the p-adic étale cohomology groups.

In this subsection, we illustrate how the classical Hodge theory inspires fundamental results in p-adic Hodge theory which relates the p-adic étale cohomology groups of an algebraic variety over $\mathbb{Q}_p$ (or its finite extension) to other cohomology groups. Let us consider an elliptic curve E over $\mathbb{Q}$. We may identify $E(\mathbb{C})$ as a complex torus via an isomorphism

$$E(\mathbb{C}) \simeq \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \quad \text{for some nonreal } \tau \in \mathbb{C}.$$

Let α and β respectively denote the loops on $E(\mathbb{C})$ induced by the line segments on $\mathbb{C}$ connecting 0 to 1 and τ, as illustrated in the following figure:



We have an isomorphism

$$H_1(E(\mathbb{C}), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$$

with a basis given by the homotopy classes of α and β, and consequently find

$$H^1(E(\mathbb{C}), \mathbb{C}) \cong \text{Hom}(H_1(E(\mathbb{C}), \mathbb{C})) \simeq \mathbb{C} \oplus \mathbb{C} \tag{1.3}$$

by Poincaré duality. Moreover, since $E(\mathbb{C})$ has genus 1 there exists an isomorphism

$$H^0(E_{\mathbb{C}}, \Omega^1_{E_{\mathbb{C}}}) \simeq \mathbb{C}$$

with a basis given by dz. Hence we obtain an isomorphism

$$H^0(E_{\mathbb{C}}, \Omega^1_{E_{\mathbb{C}}}) \oplus \overline{H^0(E_{\mathbb{C}}, \Omega^1_{E_{\mathbb{C}}})} \xrightarrow{\sim} H^1(E(\mathbb{C}), \mathbb{C}) \tag{1.4}$$

which maps dz and $d\overline{z}$ respectively to $\int dz = (1, \tau)$ and $\int d\overline{z} = (1, \overline{\tau})$ under the isomorphism (1.3). It is not hard to see that this isomorphism is canonical. In fact, it is a special case of the *Hodge decomposition* given by the following theorem:

THEOREM 1.2.1. For a smooth proper variety X over $\mathbb{C}$, there exists a canonical isomorphism

$$H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^n_{\mathrm{dR}}(X/\mathbb{C}) \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_X)$$

with $\overline{H^i(X, \Omega^j_X)} = H^j(X, \Omega^i_X)$.

Theorem 1.2.1 admits analogues for the p-adic étale cohomology of an algebraic variety over $\mathbb{Q}_p$. Let $\mathbb{C}_p$ denote the p-adic completion of $\overline{\mathbb{Q}}_p$, called the field of *p-adic complex numbers*. The field $\mathbb{C}_p$ is complete and algebraically closed, just as the field $\mathbb{C}$ is. Since the tautological action of $\Gamma_{\mathbb{Q}_p}$ on $\overline{\mathbb{Q}}_p$ is continuous, it uniquely extends to an action on $\mathbb{C}_p$. For a p-adic analogue of the complex conjugate, we consider the *p-adic cyclotomic character*

$$\chi : \Gamma_{\mathbb{Q}_p} \longrightarrow \mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_p^\times$$

given by the $\Gamma_{\mathbb{Q}_p}$-action on the group

$$T_p(\mu_{p^\infty}) := \varprojlim \mu_{p^v}(\overline{\mathbb{Q}}_p) \simeq \varprojlim \mathbb{Z}/p^v\mathbb{Z} = \mathbb{Z}_p$$

where $\mu_{p^v}(\overline{\mathbb{Q}}_p)$ denotes the group of p^v-th roots of unity in $\overline{\mathbb{Q}}_p$, and write $\mathbb{C}_p(n)$ for $\mathbb{C}_p$ with $\Gamma_{\mathbb{Q}_p}$-action twisted by χ^n in the sense that each $\gamma \in \Gamma_{\mathbb{Q}_p}$ acts on $\mathbb{C}_p(n)$ as $\chi(\gamma)^n\gamma$. For an elliptic curve E over $\mathbb{Q}_p$ with good reduction, the work of Tate [**Tat67**] yields a canonical isomorphism

$$H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong H^0(E, \Omega^1_{E/\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \oplus H^1(E, \Omega^0_{E/\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(-1)$$

which is compatible with $\Gamma_{\mathbb{Q}_p}$-actions. In fact, this isomorphism is a special case of the *Hodge-Tate decomposition* given by the following theorem:

THEOREM 1.2.2 (Faltings [**Fal88**]). For a smooth proper variety X over $\mathbb{Q}_p$, there exists a canonical isomorphism

$$H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(-j) \tag{1.5}$$

which is compatible with $\Gamma_{\mathbb{Q}_p}$-actions.

Let us take the *Hodge-Tate period ring* $B_{\mathrm{HT}} := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_p(n)$ and write the isomorphism (1.5) as a $\Gamma_{\mathbb{Q}_p}$-equivariant isomorphism of graded algebras

$$H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}} \cong \Big(\bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{Q}_p}) \Big) \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}. \tag{1.6}$$

A result of Tate [**Tat67**] and Sen [**Sen80**] establishes an identification $B_{\mathrm{HT}}^{\Gamma_{\mathbb{Q}_p}} = \mathbb{Q}_p$ and in turn yields an isomorphism of graded $\mathbb{Q}_p$-algebras

$$\big(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}\big)^{\Gamma_{\mathbb{Q}_p}} \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{Q}_p}).$$

In particular, we can compute the Hodge numbers of X from $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$.

Theorem 1.2.2 is, however, not a complete analogue of Theorem 1.2.1 as it does not give a comparison isomorphism which directly relate the étale cohomology and the de Rham cohomology. Fontaine [**Fon82**] formulated a conjecture that such a comparison isomorphism exists as a refinement of the isomorphism (1.6), inspired by the fact that the de Rham cohomology group $H^n_{\text{dR}}(X/\mathbb{Q}_p)$ has a natural filtration $\left\{ \text{Fil}^m \left(H^n_{\text{dR}}(X/\mathbb{Q}_p) \right) \right\}_{m\in\mathbb{Z}}$, called the *Hodge filtration*, with its graded vector space $\text{gr}\,(H^n_{\text{dR}}(X/\mathbb{Q}_p))$ yielding a natural isomorphism

$$\text{gr}\,(H^n_{\text{dR}}(X/\mathbb{Q}_p)) \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{Q}_p}).$$

A key ingredient of the conjecture is the *de Rham period ring* B_{dR} which Fontaine [**Fon82**] constructed as a $\mathbb{Q}_p$-algebra with the following properties:

(i) B_{dR} carries a natural action of $\Gamma_{\mathbb{Q}_p}$ with $B_{\text{dR}}^{\Gamma_{\mathbb{Q}_p}} = \mathbb{Q}_p$.
(ii) B_{dR} admits a natural filtration $\{ \text{Fil}^n(B_{\text{dR}}) \}_{n\in\mathbb{Z}}$ with B_{HT} as its graded algebra.

Fontaine's conjecture is now a theorem, commonly referred to as the *p-adic de Rham comparison theorem*, which we state as follows:

THEOREM 1.2.3 (Faltings [**Fal89**]). For a smooth proper variety X over $\mathbb{Q}_p$, there exists a canonical isomorphism

$$H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \cong H^n_{\text{dR}}(X/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \tag{1.7}$$

which is compatible with $\Gamma_{\mathbb{Q}_p}$-actions and filtrations.

Remark. The filtration on the right side is the *convolution filtration* given by

$$\text{Fil}^m \left(H^n_{\text{dR}}(X/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \right) := \bigoplus_{i+j=m} \text{Fil}^i \left(H^n_{\text{dR}}(X/\mathbb{Q}_p) \right) \otimes_{\mathbb{Q}_p} \text{Fil}^j(B_{\text{dR}}) \quad \text{for every } m \in \mathbb{Z}.$$

Theorem 1.2.3 yields Theorem 1.2.2 as a formal consequence; indeed, we obtain the isomorphism (1.6) from the isomorphism (1.7) by passing to the associated graded vector spaces. In addition, Theorem 1.2.3 induces a natural isomorphism

$$\left(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \right)^{\Gamma_{\mathbb{Q}_p}} \cong H^n_{\text{dR}}(X/\mathbb{Q}_p),$$

thereby allowing us to recover $H^n_{\text{dR}}(X/\mathbb{Q}_p)$ from $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$. Therefore Theorem 1.2.3 (with Theorem 1.2.2 as its consequence) indicates that the p-adic étale cohomology of an algebraic variety over $\mathbb{Q}_p$ behaves much as the singular cohomology of an algebraic variety over $\mathbb{C}$ does.

Let us now assume that X has good reduction over $\mathbb{Q}_p$. Intuitively, our assumption means that we may regard X as a smooth scheme over $\mathbb{Z}_p$, and thus allows us to take its mod p reduction $\overline{X}$. Motivated by our discussion in §1.1, we wish to understand the p-adic Galois representation $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ using $\overline{X}$. We consider the *crystalline cohomology group* $H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p)$ which is a Dieudonné module over $\mathbb{Z}_p$ with a natural isomorphism

$$H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^n_{\text{dR}}(X/\mathbb{Q}_p)$$

and a canonical filtration $\left\{ \text{Fil}^m \left(H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \right) \right\}_{m\in\mathbb{Z}}$ induced by the Hodge filtration on $H^n_{\text{dR}}(X/\mathbb{Q}_p)$. For an elliptic curve E with good reduction over $\mathbb{Q}_p$, we may naturally identify $H^1_{\text{cris}}(\overline{E}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with the filtered isocrystal associated to E, which in turn determines $H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \cong V_p(E)^\vee$ by our discussion in §1.1. For the general case, Grothendieck [**Gro71**] proposed a conjecture that $H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as a filtered isocrystal determines $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ as a p-adic Galois representation in a functorial way; indeed, his conjecture predicts that there exists a fully faithful functor $\mathcal{D}$ on a certain category of p-adic Galois representations with

$$\mathcal{D}\big(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \big) = H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

We refer to the functor $\mathcal{D}$ as the *Grothendieck mysterious functor*.

Fontaine [**Fon82, Fon83**] reformulated the conjecture of Grothendieck [**Gro71**] in terms of a comparison isomorphism between the étale cohomology and the crystalline cohomology. His idea was to refine the de Rham comparison isomorphism (1.7) by constructing the *crystalline period ring* B_{cris}, which is a $\mathbb{Q}_p$-subalgebra of B_{dR} with the following properties:

(i) B_{cris} carries a natural action of Γ_K with $B_{\text{cris}}^{\Gamma_{\mathbb{Q}_p}} = \mathbb{Q}_p$, induced by the action on B_{dR}.
(ii) B_{cris} admits a (Frobenius-semilinear) endomorphism φ, called the *Frobenius endomorphism*, and a natural filtration $\{ \operatorname{Fil}^n(B_{\text{cris}}) \}_{n\in\mathbb{Z}}$ given by the filtration on B_{dR}.

Fontaine's conjecture is now a theorem, commonly referred to as the *crystalline comparison theorem*, which we state as follows:

THEOREM 1.2.4 (Faltings [**Fal89**]). For a smooth proper variety X over $\mathbb{Q}_p$ with mod p reduction $\overline{X}$, there exists a canonical isomorphism

$$H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \cong H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}} \tag{1.8}$$

which is compatible with $\Gamma_{\mathbb{Q}_p}$-actions, filtrations, and Frobenius actions.

Remark. As in Theorem 1.2.3, the right side carries the convolution filtration given by

$$\operatorname{Fil}^m \left(H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}}\right) := \bigoplus_{i+j=m} \operatorname{Fil}^i \left(H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p)\right) \otimes_{\mathbb{Z}_p} \operatorname{Fil}^j(B_{\text{cris}}) \quad \text{for every } m \in \mathbb{Z}.$$

The Frobenius actions refer to the Frobenius endomorphisms on $H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p)$ and B_{cris}.

Under the assumption that X has good reduction, we can obtain the de Rham comparison isomorphism (1.7) from the crystalline comparison isomorphism (1.8) by tensoring with B_{dR} and forgetting the Frobenius actions. In addition, Theorem 1.2.4 yields a natural isomorphism

$$\left(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}}\right)^{\Gamma_{\mathbb{Q}_p}} \cong H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

thereby suggesting that the mysterious functor $\mathcal{D}$ takes the form

$$\mathcal{D}(V) = (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_{\mathbb{Q}_p}}$$

for every p-adic Galois representation V. It turns out, by the work of Fontaine [**Fon94**], that the functor $\mathcal{D}$ is fully faithful on a suitable category of p-adic Galois representations with values taken in the category of filtered isocrystals. In fact, $H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ determines $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ by an identification

$$H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \cong \left(H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}}\right)^{\varphi=1} \cap \operatorname{Fil}^0 \left(H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}}\right) \tag{1.9}$$

where $\left(H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}}\right)^{\varphi=1}$ denotes the space of invariants in $H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}}$ under the Frobenius action.

As our discussion demonstrates, a main theme in p-adic Hodge theory is to establish a comparison isomorphism that relates p-adic étale cohomology groups to cohomology groups of a different kind. In addition to the theorems presented in this subsection, there are many results of a similar flavor, notably by the work of Tsuji [**Tsu99**], Scholze [**Sch13**], and Bhatt-Morrow-Scholze [**BMS18, BMS19**]. Let us also mention that there are other approaches for the comparison theorems presented in this subsection, in particular by the work of Fontaine-Messing [**FM87**], Nizioł [**Niz98, Niz08**], and Beilinson [**Bei12, Bei13**].

1.3. The interplay via algebraic functors

In the previous subsections, we illustrated two main themes in p-adic Hodge theory. The first one, from the arithmetic perspective, is to construct a dictionary that relates p-adic Galois representations to various linear algebraic objects. The second one, from the geometric perspective, is to establish a comparison isomorphism that relates p-adic étale cohomology groups to other cohomology groups.

In this subsection, we describe a connection between the two main themes of p-adic Hodge theory provided by some linear algebraic functors. These functors originate in the work of Fontaine [**Fon79, Fon82, Fon83**] which proposes a uniform approach for the p-adic comparison theorems in an attempt to resolve the conjecture of Grothendieck [**Gro71**] on the mysterious functor. We write $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p})$ for the category of p-adic Galois representations, and $\mathrm{Vect}_{\mathbb{Q}_p}$ for the category of finite dimensional vector spaces over $\mathbb{Q}_p$. Let B be a *p-adic period ring*, such as B_{HT}, B_{dR} or B_{cris}, which is a $\mathbb{Q}_p$-algebra carrying a natural $\Gamma_{\mathbb{Q}_p}$-action with $B^{\Gamma_{\mathbb{Q}_p}} = \mathbb{Q}_p$. We define the functor $D_B : \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p}) \longrightarrow \mathrm{Vect}_{\mathbb{Q}_p}$ by setting

$$D_B(V) := (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_{\mathbb{Q}_p}} \qquad \text{for each } V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p})$$

and say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p})$ is *B-admissible* if the natural $\Gamma_{\mathbb{Q}_p}$-equivariant map

$$\alpha_V : D_B(V) \otimes_{\mathbb{Q}_p} B \longrightarrow (V \otimes_{\mathbb{Q}_p} B) \otimes_{\mathbb{Q}_p} B \cong V \otimes_{\mathbb{Q}_p} (B \otimes_{\mathbb{Q}_p} B) \longrightarrow V \otimes_{\mathbb{Q}_p} B$$

is an isomorphism. We enhance the functor D_B by incorporating additional structures on B, as demonstrated by the following examples:

(1) $D_{B_{\mathrm{HT}}}(V)$ for each $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p})$ carries a grading naturally induced by the grading on B_{HT}.

(2) $D_{B_{\mathrm{dR}}}(V)$ for each $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p})$ carries a filtration naturally induced by the filtration on B_{dR}.

(3) $D_{B_{\mathrm{cris}}}(V)$ for each $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p})$ carries a Frobenius endomorphism and a filtration naturally induced by the ones on B_{cris}.

Then for a smooth proper variety X over $\mathbb{Q}_p$, we may state the p-adic comparison theorems from §1.2 as follows:

(1) $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is B_{HT}-admissible with a natural isomorphism

$$D_{B_{\mathrm{HT}}}(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)) \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{Q}_p})$$

which is compatible with gradings on both sides.

(2) $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is B_{dR}-admissible with a natural isomorphism

$$D_{B_{\mathrm{dR}}}(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)) \cong H^n_{\mathrm{dR}}(X/\mathbb{Q}_p)$$

which is compatible with filtrations on both sides.

(3) If X admits mod p reduction $\overline{X}$, then $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is B_{cris}-admissible with a natural isomorphism

$$D_{B_{\mathrm{cris}}}(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)) \cong H^n_{\mathrm{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

which is compatible with Frobenius endomorphisms and filtrations on both sides.

Let us denote by $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$ the category of B-admissible representations. The work of Fontaine [**Fon82, Fon83**] yields a hierarchy of p-adic Galois representations given by

$$\mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{cris}}}(\Gamma_{\mathbb{Q}_p}) \subsetneq \mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{dR}}}(\Gamma_{\mathbb{Q}_p}) \subsetneq \mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{HT}}}(\Gamma_{\mathbb{Q}_p})$$

with the associated functors satisfying the following relations:

- $D_{B_{\mathrm{HT}}}(V)$ for each $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{dR}}}(\Gamma_{\mathbb{Q}_p})$ is naturally isomorphic to the graded vector space of $D_{B_{\mathrm{dR}}}(V)$.
- $D_{B_{\mathrm{dR}}}(V)$ for each $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{cris}}}(\Gamma_{\mathbb{Q}_p})$ is naturally isomorphic to $D_{B_{\mathrm{cris}}}(V)$ (after forgetting the Frobenius endomorphism).

This hierarchy realizes relations between various cohomology groups for a smooth proper variety X over $\mathbb{Q}_p$, as presented in §1.2 and summarized in the following statements:

- The Hodge-Tate decomposition (1.6) follows from the de Rham comparison isomorphism (1.7) by passing to the associated graded space via the identification
$$\mathrm{gr}\,(H^n_{\mathrm{dR}}(X/\mathbb{Q}_p)) \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{Q}_p}).$$
where $\mathrm{gr}\,(H^n_{\mathrm{dR}}(X/\mathbb{Q}_p))$ denote the graded vector space of $H^n_{\mathrm{dR}}(X/\mathbb{Q}_p)$.
- If X has good reduction, the de Rham comparision isomorphism (1.7) follows from the crystalline comparison isomorphism (1.8) by tensoring with B_{dR} via the identification
$$H^n_{\mathrm{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^n_{\mathrm{dR}}(X/\mathbb{Q}_p).$$

In fact, we can conceptualize our hierarchy by the following principles:

(1) $\mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{HT}}}(\Gamma_{\mathbb{Q}_p})$ contains almost all p-adic Galois representations which arise in practice.
(2) $\mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{dR}}}(\Gamma_{\mathbb{Q}_p})$ contains all p-adic Galois representations which come from geometry.
(3) $\mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{cris}}}(\Gamma_{\mathbb{Q}_p})$ contains all p-adic Galois representations which come from geometry with integral structures.

We wish to understand how the category $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$ behaves, especially in conjunction with the functor D_B. A general formalism developed by Fontaine [**Fon94**] shows that $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$ and D_B have the following properties:

(i) D_B is exact and faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$.
(ii) $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$ is closed under taking subquotients.
(iii) $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$ is closed under tensor products, with a natural identification
$$D_B(V \otimes_{\mathbb{Q}_p} W) \cong D_B(V) \otimes_{\mathbb{Q}_p} D_B(W) \quad \text{for any } V, W \in \mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p}).$$
(iv) $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$ is closed under taking duals, with a natural identification
$$D_B(V^\vee) \cong \mathrm{Hom}_{\mathbb{Q}_p}(D_B(V), \mathbb{Q}_p) \quad \text{for every } V \in \mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$$
where $V^\vee$ denotes the dual representation of V.

Moreover, $D_{B_{\mathrm{cris}}}$ and $\mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{cris}}}(\Gamma_{\mathbb{Q}_p})$ have a remarkable property given by the following result:

THEOREM 1.3.1 (Fontaine [**Fon94**]). *The functor $D_{B_{\mathrm{cris}}}$ is fully faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{cris}}}(\Gamma_{\mathbb{Q}_p})$.*

Remark. Theorem 1.2.4 and Theorem 1.3.1 together resolve the conjecture of Grothendieck [**Gro71**] on the mysterious functor.

Theorem 1.3.1 implies that studying B_{cris}-admissible representations is equivalent to studying their associated filtered isocrystals. Therefore it is vital to understand the essential image of $D_{B_{\text{cris}}}$, called the category of *admissible filtered isocrystals* over $\mathbb{Q}_p$. To every filtered isocrystal over $\mathbb{Q}_p$, we attach two invariants called the *Newton polygon* and the *Hodge polygon* , which are convex polygons with integer breakpoints. By definition, the Newton polygon encodes the eigenspace decomposition for the Frobenius endomorphism while the Hodge polygon encodes the isomorphism class of the associated graded vector space. A remarkable result of Mazur [**Maz72**] and Berthelot-Ogus [**BO78**] is that for a smooth proper variety X over $\mathbb{Q}_p$ with mod p reduction $\overline{X}$ the Newton polygon of $H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes \mathbb{Q}_p$ lies on or above the Hodge polygon of $H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes \mathbb{Q}_p$ with same endpoints. Inspired by this result, the work of Colmez-Fontaine [**CF00**] provides an intrinsic description for the category of admissible isocrystals by some explicit conditions on Newton polygons and Hodge polygons as follows:

THEOREM 1.3.2 (Colmez-Fontaine [**CF00**]). *A filtered isocrystal N over $\mathbb{Q}_p$ is admissible if and only if it satisfies the following properties:*

(i) *For every filtered isocrystal $M \subseteq N$, its Newton polygon lies above its Hodge polygon.*
(ii) *The Newton polygon and the Hodge polygon of N have the same endpoints.*

The functor $D_{B_{\text{cris}}}$ and the notion of B_{cris}-admissibility are very useful for studying elliptic curves. A key strategy is, as already demonstrated in §1.1, to obtain information about the p-adic Tate module of an elliptic curve over $\mathbb{Q}$ from the Frobenius action and the filtration on the associated filtered isocrystal. As an application of this strategy, we can show that for an elliptic curve E over $\mathbb{Q}$ with mod p reduction $\overline{E}$ the Newton polygon and the Hodge polygon of $D_{B_{\text{cris}}}(V_p(E))$ coincide if and only if $V_p(\overline{E})$ has dimension 1. For another application, we have a p-adic analogue of Theorem 1.1.2 given by the following result:

THEOREM 1.3.3 (Coleman-Iovita [**CI99**]). *An elliptic curve E over $\mathbb{Q}$ has good reduction at p if and only if $T_p(E)$ is B_{cris}-admissible.*

Remark. Both Theorem 1.1.2 and Theorem 1.3.3 readily extend to *abelian varieties*, which are projective varieties with a (commutative) group structure on the set of points.

In order to study $\text{Rep}_{\mathbb{Q}_p}^{B_{\text{HT}}}(\Gamma_{\mathbb{Q}_p})$, the largest category in our hierarchy of p-adic Galois representations, we often consider invariants called *Hodge-Tate weights*. By definition, an integer d is a Hodge-Tate weight of $V \in \text{Rep}_{\mathbb{Q}_p}^{B_{\text{HT}}}(\Gamma_{\mathbb{Q}_p})$ with multiplicity m if and only if the degree d part of the graded vector space $D_{B_{\text{HT}}}(V)$ has dimension m. Hodge-Tate weights are essentially algebraic generalizations of Hodge numbers; indeed, for a smooth proper variety X over $\mathbb{Q}_p$, computing its Hodge numbers is equivalent to computing the Hodge-Tate weights of its p-adic étale cohomology (with multiplicity). Moreover, Hodge-Tate weights are useful for studying B_{HT}-admissible representations which do not necessarily come from geometry; for example, a continuous character $\eta : \Gamma_{\mathbb{Q}_p} \longrightarrow \mathbb{Q}_p$ is B_{HT}-admissible with Hodge-Tate weight d if and only if the image of $I_{\mathbb{Q}_p}$ under $\eta\chi^{-d}$ is finite.

Our discussion in this subsection indicates that period rings and their associated functors provide a general framework for the two main themes in p-adic Hodge theory. From the arithmetic perspective, they provide dictionaries for classifying and studying p-adic Galois representations in terms of linear algebraic objects. From the geometric perspective, they allow us to uniformly formulate p-adic comparison theorems and to systemically detect geometric properties of an algebraic variety over $\mathbb{Q}_p$ from its p-adic étale cohomology. Therefore period rings and their associated functors are essential for studying p-adic Hodge theory via the interplay between the arithmetic and geometric perspectives.

2. A first glimpse of the Fargues-Fontaine curve

In this section, we provide a brief introduction to a remarkable geometric object called the *Fargues-Fontaine curve*, which plays a fundamental role in modern p-adic Hodge theory. The main points that we convey in this section are as follows:

(1) The Fargues-Fontaine curve is akin to the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ in many aspects.
(2) The Fargues-Fontaine curve provides a geometric framework for studying many important objects in p-adic Hodge theory.

Along the way, we discuss some additional facts about p-adic period rings and related objects.

2.1. Construction and basic properties

In this subsection, we demonstrate the construction and some key features of the Fargues-Fontaine curve via comparisons with the complex projective line $\mathbb{P}^1_{\mathbb{C}}$. Let us recall that $\mathbb{P}^1_{\mathbb{C}}$ has the following properties:

(i) It is noetherian, connected, and regular of dimension 1.
(ii) Its Picard group $\text{Pic}(\mathbb{P}^1_{\mathbb{C}})$ is canonically isomorphic to $\mathbb{Z}$.
(iii) It has arithmetic genus 0 in the sense that $H^1(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}})$ vanishes.
(iv) It admits a closed point ∞, namely the point at infinity, with natural isomorphisms
$$\mathbb{P}^1_{\mathbb{C}} - \infty \cong \text{Spec}\,(\mathbb{C}[z]) \quad \text{and} \quad \widehat{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}},\infty}} \cong \mathbb{C}[[z^{-1}]]$$
where $\widehat{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}},\infty}}$ denotes the completed local ring at ∞.

Property (iv) is closely related to the natural exact sequence
$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}[z] \longrightarrow \mathbb{C}((z^{-1}))/\mathbb{C}[[z^{-1}]] \longrightarrow 0. \tag{2.1}$$
Intuitively, this exact sequence indicates that we can construct $\mathbb{P}^1_{\mathbb{C}}$ by gluing the complex affine line $\mathbb{A}^1_{\mathbb{C}} = \text{Spec}\,(\mathbb{C}[z])$ to the infinitesimal disk at ∞, given by $\text{Spec}\,(\mathbb{C}[[z^{-1}]])$, along the punctured infinitesimal disk at ∞, given by $\text{Spec}\,(\mathbb{C}((z^{-1})))$.

The construction of the Fargues-Fontaine curve stems from a remarkable discovery of Fontaine [**Fon94**] that the exact sequence (2.1) admits an analogue for p-adic period rings. By construction, the de Rham period ring B_{dR} is a discretely valued complete field with residue field $\mathbb{C}_p$. We write B_{dR}^+ for the valuation ring of B_{dR} and $B_e := B_{\text{cris}}^{\varphi=1}$ for the ring of φ-invariants in B_{cris}.

THEOREM 2.1.1 (Fontaine [**Fon94**]). The natural sequence
$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow B_{\text{dR}}/B_{\text{dR}}^+ \longrightarrow 0 \tag{2.2}$$
is exact.

Remark. Theorem 2.1.1 is one of the most fundamental results in p-adic Hodge theory, with many important applications including Theorem 1.3.1 and Theorem 1.3.2.

The exact sequences (2.1) and (2.2) have the following similarities:

(1) $\mathbb{C}[[z^{-1}]]$ and B_{dR}^+ are both complete discrete valuation rings, with fraction fields respectively given by $\mathbb{C}((z^{-1}))$ and B_{dR}.
(2) $\mathbb{C}[z]$ and B_e are both principal ideal domains.

The second similarity is another surprising discovery of Fontaine, primarily based on the work of Berger [**Ber08**]. The similarities of the exact sequences (2.1) and (2.2) inspire the construction of the Fargues-Fontaine curve X by gluing $\text{Spec}\,(B_e)$ and $\text{Spec}\,(B_{\text{dR}}^+)$ along $\text{Spec}\,(B_{\text{dR}})$.

THEOREM 2.1.2 (Fargues-Fontaine [**FF18**]). The Fargues-Fontaine curve X is a $\mathbb{Q}_p$-scheme with the following properties:

(i) It is noetherian, connected and regular of dimension 1.
(ii) Its Picard group $\mathrm{Pic}(X)$ is canonically isomorphic to $\mathbb{Z}$.
(iii) It has arithmetic genus 0 in the sense that $H^1(X, \mathcal{O}_X)$ vanishes.
(iv) It admits a closed point ∞ with natural isomorphisms

$$X - \infty \cong \mathrm{Spec}\,(B_e) \quad \text{and} \quad \widehat{\mathcal{O}_{X,\infty}} \cong B_{\mathrm{dR}}^+$$

where $\widehat{\mathcal{O}_{X,\infty}}$ denotes the completed local ring at ∞.

Remark. However, unlike $\mathbb{P}^1_{\mathbb{C}}$, the Fargues-Fontaine curve is not an algebraic variety. The main issue is that it is not of finite type over the base field $\mathbb{Q}_p$; indeed, property (iv) implies that the residue field at ∞ is $\mathbb{C}_p$ and thus is not finitely generated over $\mathbb{Q}_p$.

For an explicit description of the Fargues-Fontaine curve, we have a natural isomorphism $X \cong \mathrm{Proj}\,(P)$ for a graded ring

$$P := \bigoplus_{n \geq 0} B_e^{(n)}$$

where we set $B_e^{(n)} := \{ f \in B_e : \nu_\infty(f) \geq -n \}$ with ν_∞ denoting the valuation on B_{dR}. For comparison, we have the identification $\mathbb{P}^1_{\mathbb{C}} = \mathrm{Proj}\,(\mathbb{C}[z_0, z_1])$ and an isomorphism

$$\mathbb{C}[z_0, z_1] \cong \bigoplus_{n \geq 0} \mathbb{C}[z]^{(n)}$$

where we set $\mathbb{C}[z]^{(n)} := \{ f \in \mathbb{C}[z] : \nu_\infty(f) \geq -n \} = \{ f \in \mathbb{C}[z] : \deg(f) \leq n \}$ with ν_∞ denoting the valuation on $\mathbb{C}((z^{-1}))$. The graded rings P and $\mathbb{C}[z_0, z_1]$ have an important common feature of being generated in degree 1 (i.e., being generated by elements in $B_e^{(1)}$ and $\mathbb{C}[z]^{(1)}$). In fact, this feature is responsible for numerous similarities between X and $\mathbb{P}^1_{\mathbb{C}}$.

The Fargues-Fontaine curve has a surprising connection to *perfectoid fields*, which are nonarchemedan fields of a special kind introduced by Scholze [**Sch12**]. Perfectoid fields are very useful for studying problems in characteristic 0 by converting them to problems in positive characteristic. The key underlying fact is that every perfectoid field C with residue characteristic p gives rise to a perfectoid field in characteristic p given by

$$C^\flat := \varprojlim_{x \mapsto x^p} C,$$

called the *tilt* of C. For example, $\mathbb{C}_p$ is a perfectoid field with its tilt $F := \mathbb{C}_p^\flat$ naturally isomorphic to the completion of $\overline{\mathbb{F}_p((t))}$. Let us consider the set $\widehat{Y}$ of *untilts* of F, which refer to equivalence classes of pairs consisting of a perfectoid field C and an isomorphism $\iota : C^\flat \simeq F$. We write o for the *trivial untilt* given by F and its identity map, which represents the unique untilt of F in characteristic p. The set $Y := \widehat{Y} - o$ admits a natural action of the Frobenius automorphism φ_F on F given by mapping each $(C, \iota) \in Y$ to $(C, \varphi_F \circ \iota)$. By a result of Kedlaya-Liu [**KL15**], the set $|X|$ of closed points on X admits a natural bjection

$$|X| \xrightarrow{\sim} Y/\varphi_F^{\mathbb{Z}} = \big(\widehat{Y} - o\big)/\varphi_F^{\mathbb{Z}} \tag{2.3}$$

where $\varphi_F^{\mathbb{Z}}$ denotes the cyclic group generated by φ_F. We note that this bijection is reminiscent of the isomorphism $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C}) \cong \big(\mathbb{C}^2 - (0,0)\big)/\mathbb{C}^\times$ with $\mathbb{C}^\times$ acting by scalar multiplication.

2.2. Vector bundles and p-adic Galois representations

The construction of the Fargues-Fontaine curve manifests direct links to p-adic Hodge theory. In particular, it provides a geometric description for several rings in p-adic Hodge theory and encodes a remarkable relation between these rings given by Theorem 2.1.1. The work of Fargues-Fontaine [**FF18**] greatly extend these links to incorporate many other objects in p-adic Hodge theory using vector bundles on X (i.e., locally free sheaves of finite rank).

In this subsection, we illustrate the significance of vector bundles on X in p-adic Hodge theory, with particular focus on their relation to isocrystals and p-adic Galois representations. As a key technical result, the work of Fargues-Fontaine [**FF18**] establishes a classification theorem for vector bundles on X. Let us recall that, by a celebrated theorem of Grothendieck [**Gro57**], every vector bundle $\mathcal{V}$ on $\mathbb{P}^1_{\mathbb{C}}$ admits a direct sum decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d_i) \qquad \text{with } d_i \in \mathbb{Z}$$

where $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d_i)$ denotes the line bundle on $\mathbb{P}^1_{\mathbb{C}}$ corresponding to d_i under the isomorphism $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}}) \cong \mathbb{Z}$. The classification theorem for vector bundles on X yields an analogous decomposition, although the direct summands are not necessarily line bundles. For a precise statement, we define the *degree* of a vector bundle $\mathcal{V}$ on X to be the image of $\det(\mathcal{V}) := \wedge^{\mathrm{rk}(\mathcal{V})}\mathcal{V}$ under the isomorphism $\mathrm{Pic}(X) \cong \mathbb{Z}$, where $\mathrm{rk}(\mathcal{V})$ denotes the rank of $\mathcal{V}$.

THEOREM 2.2.1 (Fargues-Fontaine [**FF18**]). We can classify the vector bundles on X as follows:

(1) For a rational number $\lambda = d/r$ written in a reduced form with positive denominator, there exists a unique indecomposable vector bundle $\mathcal{O}_X(\lambda)$ of rank r and degree d

(2) Every vector bundle $\mathcal{V}$ on X admits a direct sum decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{m} \mathcal{O}_X(\lambda_i) \qquad \text{with } \lambda_i \in \mathbb{Q}.$$

Remark. Kedlaya [**Ked04, Ked05**] obtained an equivalent statement of Theorem 2.2.1 prior to the work of Fargues-Fontaine [**FF18**]. His result concerns certain analogues of isocrystals and leads to a number of important results for studying the Fargues-Fontaine curve.

Theorem 2.2.1 finds its motivation in an analogous classification theorem for isocrystals. Let us denote the completion of $\mathbb{Q}_p^{\mathrm{un}}$ by $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$. The isocrystals over $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ of rank 1 are canonically in bijection with the integers, where each isocrystal N over $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ of rank 1 maps to the p-adic valuation of $\varphi_N(1)$ upon choosing an isomorphism $N \simeq \widehat{\mathbb{Q}_p^{\mathrm{un}}}$. We define the *degree* of an isocrystal N over $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ to be the integer corresponding to the isocrystal $\det(N) := \wedge^{\mathrm{rk}(N)}N$ over $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ of rank 1, where $\mathrm{rk}(N)$ denotes the rank of N.

THEOREM 2.2.2 (Manin [**Man63**]). We can classify the isocrystals over $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ as follows:

(1) For a rational number $\lambda = d/r$ written in a reduced form with positive denominator, there exists a unique simple isocrystal $N(\lambda)$ over $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ of rank r and degree d.

(2) Every isocrystal N over $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ admits a direct sum decomposition

$$N \simeq \bigoplus_{i=1}^{m} N(\lambda_i) \qquad \text{with } \lambda_i \in \mathbb{Q}.$$

In fact, the work of Fargues-Fontaine [**FF18**] reveals a tidy connection between the category Bun_X of vector bundles on X and the category $\varphi-\mathrm{Mod}_{\mathbb{Q}_p}$ of isocrystals over $\mathbb{Q}_p$, given by an essentially surjective functor

$$\mathcal{E} : \varphi-\mathrm{Mod}_{\mathbb{Q}_p} \longrightarrow \mathrm{Bun}_X$$

which is compatible with ranks, degrees, direct sums, and tensor products. The key observation is that we can produce a vector bundle $\mathcal{V}$ on X by gluing a vector bundle $\mathcal{V}^\circ$ on $\mathrm{Spec}\,(B_e)$ to a vector bundle $\widehat{\mathcal{V}_\infty}$ on $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$ along $\mathrm{Spec}\,(B_{\mathrm{dR}})$; in other words, we obtain a vector bundle on X from a pair $(M^\circ, \widehat{M_\infty})$ consisting of a free B_e-module M° of finite rank and a B_{dR}^+-lattice $\widehat{M_\infty}$ in $M^\circ \otimes_{B_e} B_{\mathrm{dR}}$. The functor $\mathcal{E}$ sends each isocrystal N over $\mathbb{Q}_p$ to the vector bundle obtained from the pair $(N^{\varphi=1} \otimes_{\mathbb{Q}_p} B_e, N \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)$ where $N^{\varphi=1}$ denotes the space of φ_N-invariants in N.

On the category $\mathrm{MF}_{\mathbb{Q}_p}^{\varphi}$ of filtered isocrystals over $\mathbb{Q}_p$, we have another functor

$$\mathcal{E}' : \mathrm{MF}_{\mathbb{Q}_p}^{\varphi} \longrightarrow \mathrm{Bun}_X$$

which sends each filtered isocrystal N over $\mathbb{Q}_p$ with filtration $\{\, \mathrm{Fil}^n(N) \,\}_{n \in \mathbb{Z}}$ to the vector bundle obtained from the pair $(N^{\varphi=1} \otimes_{\mathbb{Q}_p} B_e, \mathrm{Fil}^0(N \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}))$ with

$$\mathrm{Fil}^0(N \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Fil}^n(N) \otimes_{\mathbb{Q}_p} \mathrm{Fil}^{-n}(B_{\mathrm{dR}}).$$

The vector bundle $\mathcal{E}'(N)$ for each $N \in \mathrm{MF}_{\mathbb{Q}_p}^{\varphi}$ carries a natural action of $\Gamma_{\mathbb{Q}_p}$ induced by the $\Gamma_{\mathbb{Q}_p}$-action on B_{dR}, as the ring B_e and the filtration on B_{dR} turn out to be stable under the $\Gamma_{\mathbb{Q}_p}$-action on B_{dR}. The functor $\mathcal{E}'$ allows us to study filtered isocrystals and p-adic Galois representations via vector bundles on X, as indicated by the following facts:

(1) There exists a natural $\Gamma_{\mathbb{Q}_p}$-equivariant isomorphism

$$V \cong H^0(X, \mathcal{E}'(D_{B_{\mathrm{cris}}}(V))) \quad \text{for every } V \in \mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{cris}}}(\Gamma_{\mathbb{Q}_p}).$$

(2) Every $N \in \mathrm{MF}_{\mathbb{Q}_p}^{\varphi}$ is admissible if and only if $\mathcal{E}'(N)$ is trivial.

The first fact follows directly from the constructions of X and $\mathcal{E}'$, whereas the second fact is a consequence of Theorem 2.2.1. It is worthwhile to mention that these facts yield geometric proofs of Theorem 1.3.1 and Theorem 1.3.2.

Let us finish this section by addressing another major application of the Fargues-Fontaine, whose scope reaches far beyond p-adic Hodge theory. One of the most influential research projects in modern mathematics is the *Langlands program*, which investigates intricate connections between various areas of mathematics, such as number theory, geometry, and complex analysis. The Fargues-Fontaine curve has a remarkable application to a central conjecture in the Langlands program, namely the *local Langlands correspondence*, which aims to relate representations of algebraic groups over $\mathbb{Q}_p$ to representations of $\Gamma_{\mathbb{Q}_p}$. In fact, the seminal work of Fargues-Scholze [**FS21**] proposes a geometric construction of the local Langlands correspondence in terms of vector bundles on the Fargues-Fontaine curve. The construction involves vast generalizations of several facts presented in this section, including the bijection (2.3) and Theorem 2.2.1, in addition to a number of advanced tools from p-adic geometry. While we are unable to discuss any details about the construction in this book, we hope that our brief exhibition inspires curious readers to study related topics. The book of Scholze-Weinstein [**SW20**] is a wonderful introductory reference for the theoretical foundations.

Exercises

1. Let E be an elliptic curve over $\mathbb{Q}$, defined by an equation

$$y^2 = x^3 + ax + b \quad \text{with } a,\ b \in \mathbb{Q} \text{ and } 4a^3 + 27b^2 \neq 0.$$

(1) Show that every nonvertical line intersects with E at three points (over $\overline{\mathbb{Q}}$), counted with multiplicity.

(2) The group law on E, written additively, satisfies the following properties:

(i) The identity element O is the point at infinity.

(ii) Given a point P on E, the vertical line passing through it and E have the second intersection point at $-P$.

(iii) Given two points $P,\ Q$ on E with distinct x-coordinates, the line passing through them and E have the third intersection point at $-(P+Q)$.

(iv) Given a point P on E, the tangent line to E at P and E have the third intersection point at $-(P+P)$.

Given two arbitrary points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ on E, derive a formula for their sum $P + Q$.

Remark. The conclusions of this exercise remains valid if one replaces the base field $\mathbb{Q}$ with another field k. In addition, one can verify that the group law on E given by the above properties is indeed associative. For curious readers who attempt to check this by themselves, there are two possible approaches as follows:

(a) One can use the formula for the group law obtained here for a direct verification.

(b) One can use the Riemann-Roch theorem to show that the group law on E agrees with the group law on $\mathrm{Pic}^0(E)$, the degree 0 part of the Picard group $\mathrm{Pic}(E)$.

2. Let E be an elliptic curve over $\mathbb{Q}$ and n be a positive integer.

(1) Show that $E[n](\overline{\mathbb{Q}})$ is an abelian group of order n^2 with a natural action of $\Gamma_{\mathbb{Q}}$.

Hint. Identify $E[n](\overline{\mathbb{Q}})$ as a solution set of polynomials with rational coefficients.

(2) Establish an identification $E[n](\overline{\mathbb{Q}}) \cong (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$.

Hint. Apply the fundamental theorem for finitely generated abelian groups after observing that $E[d](\overline{\mathbb{Q}})$ has d^2 elements for each divisor d of n.

Remark. If one replaces the base field $\mathbb{Q}$ with another field k, the conclusions of this exercise remains valid as long as n is invertible in k. If n is not invertible in k, the group $E[n](\overline{k})$ still carries a natural action of the absolute Galois group $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$ but may have order less than n^2.

3. Given an elliptic curve E over $\mathbb{Q}$ and a prime number ℓ, show that the ℓ-adic Tate-module $T_\ell(E)$ is a free $\mathbb{Z}_\ell$-module of rank 2 with a natural action of $\Gamma_{\mathbb{Q}}$.

Remark. If one replaces the base field $\mathbb{Q}$ with another field k, the conclusions of this exercise remains valid as long as ℓ is different from the characteristic of k. If n has characterstic ℓ, the ℓ-adic Tate-module $T_\ell(E)$ is still a free $\mathbb{Z}_\ell$-module but of rank 0 or 1.

4. In this exercise, we give a simple analogy between the complex conjugation and the p-adic cyclotomic character.

(1) Let μ_∞ denote the group of roots of unity in $\mathbb{C}$. Show that the complex conjugation naturally induces a character
$$\tilde{\chi} : \Gamma_{\mathbb{R}} \longrightarrow \mathrm{Aut}(\mathbb{R}) \cong \mathbb{R}^\times$$
with $\gamma(\zeta) = \zeta^{\tilde{\chi}(\gamma)}$ for every $\gamma \in \Gamma_{\mathbb{R}}$ and $\zeta \in \mu_\infty$.

(2) Let μ_{p^∞} denote the group of p-power roots of unity in $\overline{\mathbb{Q}}_p$. Show that the p-adic cyclotomic character χ yields the relation $\gamma(\zeta) = \zeta^{\chi(\gamma)}$ for every $\gamma \in \Gamma_{\mathbb{Q}_p}$ and $\zeta \in \mu_{p^\infty}$.

5. This exercise requires some knowledge on the étale cohomology and the Hodge theory.

(1) Directly verify the Hodge-Tate decomposition theorem for $\mathbb{P}^1$.

(2) Show that the p-adic de Rham comparison theorem fails if we replace B_{dR} by $\mathbb{C}_p$.

6. Deduce the identification (1.9) from Theorem 1.2.4 and Theorem 2.1.1.

7. Let ν_∞ denote the valuations on B_{dR} and $\mathbb{C}((z^{-1}))$.

(1) Show the identity $\deg(f) = -\nu_\infty(f)$ for every $f \in \mathbb{C}(z)$.

(2) Define the *degree* of each $f \in B_{\mathrm{dR}}$ to be $\deg(f) := -\nu_\infty(f)$. Prove the identity
$$\deg(fg) = \deg(f) + \deg(f) \qquad \text{for any } f,\ g \in B_{\mathrm{dR}}.$$

8. In this exercise, we provide a precise description of the Fargues-Fontaine curve X as a scheme that glues $\mathrm{Spec}\,(B_e)$ and $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$ along $\mathrm{Spec}\,(B_{\mathrm{dR}})$; in other words, we prove that the topological space obtained by gluing $\mathrm{Spec}\,(B_e)$ and $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$ along $\mathrm{Spec}\,(B_{\mathrm{dR}})$ is naturally a scheme. We define the degree function on B_{dR} as in Exercise 7.

(1) Under the identification $\mathbb{A}^1_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}} - \infty$, prove the identification
$$\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(U) = \begin{cases} \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(U) & \text{for any open } U \subseteq \mathbb{P}^1_{\mathbb{C}} \text{ with } \infty \notin U, \\ \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(U - \infty)^- & \text{for any open } U \subseteq \mathbb{P}^1_{\mathbb{C}} \text{ with } \infty \in U \end{cases}$$
where we set $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(U - \infty)^- := \Big\{ f \in \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(U - \infty) : \deg(f) \leq 0 \Big\}$.

(2) Let us set $X^\circ := \mathrm{Spec}\,(B_e)$ and denote by ∞ the special point of $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$. Prove that X is indeed a scheme with the structure sheaf given by
$$\mathcal{O}_X(U) = \begin{cases} \mathcal{O}_{X^\circ}(U) & \text{for any open } U \subseteq X \text{ with } \infty \notin U, \\ \mathcal{O}_{X^\circ}(U - \infty)^- & \text{for any open } U \subseteq X \text{ with } \infty \in U \end{cases}$$
where we set $\mathcal{O}_X(U - \infty)^- := \{ f \in \mathcal{O}_{X^\circ}(U - \infty) : \deg(f) \leq 0 \}$.

9. Deduce properties (i), (ii) and (iv) in Theorem 2.1.2 from the original construction of the Fargues-Fontaine curve X, given by gluing $\mathrm{Spec}\,(B_e)$ and $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$ along $\mathrm{Spec}\,(B_{\mathrm{dR}})$, and the fact that B_e is a principal ideal domain.

CHAPTER II

Foundations of p-adic Hodge theory

1. Finite flat group schemes

In this section, we develop basic theory of finite flat group schemes and discuss some of its applications to arithmetic geometry. Our primary reference for this section is the article of Tate [**Tat97**]. Throughout our discussion, all rings are commutative unless specified otherwise.

1.1. Basic definitions and properties

We begin with the notion of group schemes over a base scheme S. We usually take S to be affine and denote the base ring by R.

Definition 1.1.1. A *group scheme* over S, or an S*-group*, is an S-scheme G with maps

- $m : G \times_S G \longrightarrow G$, called the *multiplication*,
- $e : S \longrightarrow G$, called the *unit section*,
- $i : G \longrightarrow G$, called the *inverse*,

which satisfy the group axioms given by the following commutative diagrams:

(a) associativity diagram

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{\;m\times\mathrm{id}\;} & G \times_S G \\ \Big\downarrow{\scriptstyle \mathrm{id}\times m} & & \Big\downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow{\;\;m\;\;} & G \end{array}$$

(b) identity diagrams

$$\begin{array}{ccccc} G \times_S S & & \xrightarrow{\;\;\sim\;\;} & & G \\ & {\scriptstyle \mathrm{id}\times e}\searrow & & \nearrow{\scriptstyle m} & \\ & & G \times_S G & & \end{array} \qquad \begin{array}{ccccc} S \times_S G & & \xrightarrow{\;\;\sim\;\;} & & G \\ & {\scriptstyle e\times \mathrm{id}}\searrow & & \nearrow{\scriptstyle m} & \\ & & G \times_S G & & \end{array}$$

(c) inverse diagram

$$\begin{array}{ccc} G & \overset{(i,\mathrm{id})}{\underset{(\mathrm{id},i)}{\rightrightarrows}} & G \times_S G \\ \Big\downarrow & & \Big\downarrow{\scriptstyle m} \\ S & \xrightarrow{\;\;e\;\;} & G \end{array}$$

Remark. In other words, S-groups are group objects in the category of S-schemes.

LEMMA 1.1.2. A scheme G over S is a group scheme if and only if it defines a functor from the category of S-schemes to the category of groups sending each S-scheme T to $G(T)$.

PROOF. The assertion is immediate by Yoneda's lemma. □

Definition 1.1.3. Let $f : G \to H$ be a morphism between S-groups G and H.

(1) We say that f is a *homomorphism* if the induced map $f_T : G(T) \to H(T)$ for each S-scheme T is a group homomorphism.
(2) If f is a homomorphism, we define its *kernel* to be the S-group $\ker(f)$ with $\ker(f)(T)$ for each S-scheme T given by the kernel of the induced map $f_T : G(T) \to H(T)$.

Example 1.1.4. Given an S-group G and an integer n, the *multiplication by* n on G is the homomorphism $[n]_G : G \to G$ given by the n-th power map on $G(T)$ for each S-scheme T.

LEMMA 1.1.5. Let $f : G \to H$ be a morphism between S-groups G and H.

(1) The morphism f is a homomorphism if and only if it fits ito a commutative diagram

$$\begin{array}{ccc} G \times_S G & \xrightarrow{f \times f} & H \times_S H \\ \big\downarrow{\scriptstyle m_G} & & \big\downarrow{\scriptstyle m_H} \\ G & \xrightarrow{\quad f \quad} & H \end{array}$$

where m_G and m_H respectively denote the multiplications of G and H.
(2) If f is a homomorphism, its kernel $\ker(f)$ is naturally isomorphic to the fiber of f over the unit section of H.

PROOF. The assertions are straightforward to verify by Lemma 1.1.2. □

Definition 1.1.6. Let $G = \operatorname{Spec}(A)$ be an affine R-group.

(1) Its *comultiplication* is the map $\mu : A \to A \otimes_R A$ induced by the multiplication.
(2) Its *counit* is the map $\epsilon : A \to R$ induced by the unit section.
(3) Its *coinverse* is the map $\iota : A \to A$ induced by the inverse.

LEMMA 1.1.7. Let $G = \operatorname{Spec}(A)$ be an affine R-group. Its comultiplication μ, counit ϵ, and coinverse ι fit into the following commutative diagrams:

(a) coassociativity diagram

$$\begin{array}{ccc} A \otimes_R A \otimes_R A & \xleftarrow{\mu \otimes \mathrm{id}} & A \otimes_R A \\ {\scriptstyle \mathrm{id} \otimes \mu}\big\uparrow & & \big\uparrow{\scriptstyle \mu} \\ A \otimes_R A & \xleftarrow{\quad \mu \quad} & A \end{array}$$

(b) coidentity diagrams

$$\begin{array}{ccc} A \otimes_R R & \xleftarrow{\quad\sim\quad} & A \\ & {\scriptstyle \mathrm{id}\otimes\epsilon}\nwarrow \quad \swarrow{\scriptstyle \mu} & \\ & A \otimes_R A & \end{array} \qquad \begin{array}{ccc} R \otimes_R A & \xleftarrow{\quad\sim\quad} & A \\ & {\scriptstyle \epsilon\otimes\mathrm{id}}\nwarrow \quad \swarrow{\scriptstyle \mu} & \\ & A \otimes_R A & \end{array}$$

(c) coinverse diagram

$$\begin{array}{ccc} A & \overset{\iota \otimes \mathrm{id}}{\underset{\mathrm{id} \otimes \iota}{\leftleftarrows}} & A \otimes_R A \\ \big\uparrow & & \big\uparrow{\scriptstyle \mu} \\ R & \xleftarrow{\quad \epsilon \quad} & A \end{array}$$

PROOF. The assertion is evident by definition. □

Example 1.1.8. We present some important examples of affine group schemes.

(1) The *additive group over* R is the R-scheme $\mathbb{G}_a := \text{Spec}\,(R[t])$ with the natural additive group structure on $\mathbb{G}_a(B) = B$ for each R-algebra B. Its comultiplication μ, counit ϵ, and coinverse ι are determined by the identities
$$\mu(t) = t \otimes 1 + 1 \otimes t, \qquad \epsilon(t) = 0, \qquad \iota(t) = -t.$$
(2) The *multiplicative group over* R is the R-scheme $\mathbb{G}_m := \text{Spec}\,(R[t, t^{-1}])$ with the natural multiplicative group structure on $\mathbb{G}_m(B) = B^\times$ for each R-algebra B. Its comultiplication μ, counit ϵ, and coinverse ι are determined by the identities
$$\mu(t) = t \otimes t, \qquad \epsilon(t) = 1, \qquad \iota(t) = t^{-1}.$$
(3) The *n-th roots of unity* is the R-scheme $\mu_n := \text{Spec}\,(R[t]/(t^n - 1))$ with the natural multiplicative group structure on $\mu_n(B) = \{\, b \in B : b^n = 1 \,\}$ for each R-algebra B. We can regard μ_n as a closed subgroup scheme of $\mathbb{G}_m$ via the natural surjection $R[t, t^{-1}] \twoheadrightarrow R[t]/(t^n - 1)$ with comultiplication, counit, and coinverse as in (2).
(4) If R has characteristic p, we have an R-group $\alpha_p := \text{Spec}\,(R[t]/t^p)$ with the natural additive group structure on $\alpha_p(B) = \{\, b \in B : b^p = 0 \,\}$ for each R-algebra B. We can regard α_p as a closed subgroup scheme of $\mathbb{G}_a$ by via the natural surjection $R[t] \twoheadrightarrow R[t]/(t^p)$ with comultiplication, counit, and coinverse as in (1).
(5) Given an abstract group M, the *constant group scheme* on M over R is the R-scheme $\underline{M} := \coprod_{m \in M} \text{Spec}\,(R) \cong \text{Spec}\,(A)$ for $A := \prod_{m \in M} R$ with the natural group structure (induced by M) on $\underline{M}(B)$ regarded as the set of locally constant functions from $\text{Spec}\,(B)$ to M for each R-algebra B . If we identify A and $A \otimes_R A$ respectively as the rings of R-valued functions on M and $M \times M$, the comultiplication μ, counit ϵ, and coinverse ι are given by the identities
$$\mu(f)(m, m') = f(mm'), \qquad \epsilon(f) = f(1_M), \qquad \iota(f)(m) = f(m^{-1})$$
for all $f \in A$ and $m, m' \in M$, where 1_M denotes the identity element of M.

Definition 1.1.9. Given an affine R-group $G = \text{Spec}\,(A)$, we define its *augmentation ideal* to be the kernel of its counit $\epsilon : A \to R$.

PROPOSITION 1.1.10. Let G be an affine R-group.

(1) The unit section of G is a closed embedding.
(2) The kernel of an R-group homomorphism $f : H \to G$ is a closed R-subgroup of H.

PROOF. Let us write $G = \text{Spec}\,(A)$ and denote its augmentation ideal by I. The first statement is evident as we naturally identify the unit section e of G with the closed embedding $\text{Spec}\,(A/I) \hookrightarrow \text{Spec}\,(A)$. The second statement follows from the first statement after identifying $\ker(f)$ as the fiber of f over e as noted in Lemma 1.1.5. □

Remark. Proposition 1.1.10 does not hold for general group schemes which are not necessarily affine. In fact, we can show that the unit section G is a closed embedding if and only if G is separated over R.

Example 1.1.11. Given an affine R-group G, its *n-torsion subgroup* $G[n] := \ker([n]_G)$ for each integer n is a closed R-subgroup of G by Proposition 1.1.10.

Remark. We have a natural identification $\mu_n \cong \mathbb{G}_m[n]$ for each integer $n \geq 1$.

Let us now introduce the objects of main interest for this section. For the rest of this section, we assume that R is noetherian unless stated otherwise.

Definition 1.1.12. Let $G = \mathrm{Spec}\,(A)$ be an affine group scheme over R.

(1) We say that G is *commutative* if it yields the commutative diagram

$$\begin{array}{ccc} G \times_R G & \xrightarrow{(g,h)\mapsto(h,g)} & G \times_R G \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & G & \end{array}$$

where m denotes the multiplication of G.

(2) We say that G is *finite flat of order n* if it is commutative with A being locally free of rank n over R.

Lemma 1.1.13. Let $G = \mathrm{Spec}\,(A)$ be an affine group scheme over R.

(1) G is commutative if and only if $G(B)$ is commutative for each R-algebra B.
(2) G is finite flat if and only if it is commutative with its structure morphism to $\mathrm{Spec}\,(R)$ being finite flat.

Proof. The first assertion is an immediate consequence of Lemma 1.1.2. The second assertion follows from a general fact stated in the Stacks Project [**Sta**, Tag 02KB]. □

Example 1.1.14. Some group schemes introduced in Example 1.1.8 are finite flat, as easily seen by their affine descriptions.

(1) The n-th roots of unity μ_n is finite flat of order n.
(2) If R is has characteristic p, the R-group α_p is finite flat of order p.
(3) For an abelian group M of order n, the constant R-group $\underline{M}$ is finite flat of order n.

Proposition 1.1.15. For an abelian scheme $\mathcal{A}$ of dimension g over R, its n-torsion subgroup $\mathcal{A}[n] = \ker([n]_{\mathcal{A}})$ is a finite flat R-group of order n^{2g}.

Proof. Since all fibers of $\mathcal{A}$ are abelian varieties of dimension g, the assertion follows from a standard fact about abelian varieties stated in the Stacks Project [**Sta**, Tag 03RP]. □

Remark. Readers who are unfamiliar with abelian schemes should not be concerned. For most parts of our discussion, it suffices to understand them as generalizations of elliptic curves.

Many basic properties of finite abelian groups extend to finite flat group schemes. Here we state two fundamental theorems without proof.

Theorem 1.1.16 (Deligne). Let G be a finite flat R-group of order n. The homomorphism $[n]_G$ annihilates G; in other words, it factors through the unit section of G.

Remark. Curious reader can find Deligne's proof of Theorem 1.1.16 in the lecture notes of Stix [**Sti**, §3.3]. It is unknown whether Theorem 1.1.16 holds without the commutativity assumption on G.

Theorem 1.1.17 (Grothendieck [**Gro60**]). Let G be a finite flat R-group of order n with a finite flat closed R-subgroup H of order m. There exists a unique finite flat R-group G/H of order n/m which fits into a short exact sequence

$$\underline{0} \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow \underline{0}.$$

Definition 1.1.18. Let G be a finite flat R-group with a finite flat closed R-subgroup H. We refer to the R-group G/H in Theorem 1.1.17 as the *quotient group scheme of G by H*.

1.2. Cartier duality

In this subsection, we discuss a duality for finite flat R-groups. Given an R-module M, we write $M^\vee$ for its dual module. For an R-module map f, we denote its dual map by $f^\vee$.

LEMMA 1.2.1. Let B be an R-algebra.

(1) Given an R-group G, the B-scheme G_B is naturally a B-group.
(2) Given a finite flat R-group G of order n, the B-group G_B is finite flat of order n.
(3) Given a short exact sequence of finite flat R-groups

$$\underline{0} \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow \underline{0},$$

the base change to B yields a short exact sequence

$$\underline{0} \longrightarrow (G')_B \longrightarrow G_B \longrightarrow (G'')_B \longrightarrow \underline{0}.$$

PROOF. The assertions are straightforward to verify by Lemma 1.1.2, Lemma 1.1.13, and a standard fact about finite flat morphisms stated in the Stacks project [**Sta**, Tag 02KD]. □

Definition 1.2.2. Given a finite flat R-group G, its *Cartier dual* $G^\vee$ is the group-valued functor on the category of R-algebras with

$$G^\vee(B) = \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B) \qquad \text{for each } R\text{-algebra } B$$

where the group structure is induced by the multiplication map on $(\mathbb{G}_m)_B$.

LEMMA 1.2.3. Given be a finite flat R-group G with $[n]_G = 0$, we have

$$G^\vee(B) \cong \mathrm{Hom}_{B\text{-grp}}(G_B, (\mu_n)_B) \qquad \text{for each } R\text{-algebra } B.$$

PROOF. The assertion follows immediately from the identification $\mu_n = \mathbb{G}_m[n]$. □

THEOREM 1.2.4 (Cartier duality). Let $G = \mathrm{Spec}\,(A)$ be a finite flat R-group of order n with comultiplication μ, counit ϵ, and coinverse ι. For the R-algebra A we write $s : R \to A$ for its structure morphism and $m_A : A \otimes_R A \to A$ for its ring multiplication map.

(1) $A^\vee$ is an R-algebra with structure morphism $\epsilon^\vee$ and ring multiplication map $\mu^\vee$.
(2) $G^\vee$ is an R-group which admits a natural identification $G^\vee \cong \mathrm{Spec}\,(A^\vee)$ with comultiplication $m_A^\vee$, counit $s^\vee$, and coinverse $\iota^\vee$.
(3) $G^\vee$ is finite flat of order n.
(4) There exists a canonical R-group isomorphism $G \cong (G^\vee)^\vee$.

PROOF. Let us consider the natural identifications

$$R^\vee \cong R \quad \text{ and } \quad (A \otimes_R A)^\vee \cong A^\vee \otimes_R A^\vee.$$

The map $\mu^\vee$ fits into associativity and commutativity diagrams induced by the corresponding diagrams for the multiplication on G. In addition, we have commutative diagrams

$$\begin{array}{ccc} A^\vee \otimes_R R & \xrightarrow{\sim} & A^\vee \\ & {\scriptstyle \mathrm{id}\otimes\epsilon^\vee} \searrow \quad \nearrow {\scriptstyle \mu^\vee} & \\ & A^\vee \otimes_R A^\vee & \end{array} \qquad \begin{array}{ccc} R \otimes_R A^\vee & \xrightarrow{\sim} & A^\vee \\ & {\scriptstyle \epsilon^\vee\otimes\mathrm{id}} \searrow \quad \nearrow {\scriptstyle \mu^\vee} & \\ & A^\vee \otimes_R A^\vee & \end{array}$$

induced by the identity diagrams for G. Hence we deduce statement (1).

Let us now consider statement (2). It is straightforward to verify that $G^\triangledown := \mathrm{Spec}\,(A^\vee)$ is an R-group with comultiplication $m_A^\vee$, counit $s^\vee$, and coinverse $\iota^\vee$. Let B be an arbitrary R-algebra. In light of Lemma 1.1.2, we wish to establish a canonical isomorphism

$$G^\vee(B) \cong G^\triangledown(B). \tag{1.1}$$

Let μ_B, ϵ_B, and ι_B respectively denote the comultiplication, counit, and coinverse of $G_B \cong \mathrm{Spec}\,(A_B)$. By the affine description of $\mathbb{G}_m$ given in Example 1.1.8, we find

$$G^\vee(B) = \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B) \cong \left\{ f \in \mathrm{Hom}_{B\text{-alg}}(B[t, t^{-1}], A_B) : \mu_B(f(t)) = f(t) \otimes f(t) \right\}$$

where the identity $\mu_B(f(t)) = f(t) \otimes f(t)$ comes from compatibility with comultiplications. Since we have the canonical isomorphism $\mathrm{Hom}_{B\text{-alg}}(B[t, t^{-1}], A_B) \cong A_B^\times$ which sends each $f \in \mathrm{Hom}_{B\text{-alg}}(B[t, t^{-1}], A_B)$ to $f(t)$, we obtain a natural identification

$$G^\vee(B) \cong \left\{ u \in A_B^\times : \mu_B(u) = u \otimes u \right\}. \tag{1.2}$$

Meanwhile, as $A_B^\vee$ is a B-algebra by statement (1), we have

$$G^\triangledown(B) \cong \mathrm{Hom}_{R\text{-alg}}(A^\vee, B) \cong \mathrm{Hom}_{B\text{-alg}}(A_B^\vee, B). \tag{1.3}$$

Let us denote the ring multiplication map on B by m_B and the identity map on B by id_B. By definition, $\mathrm{Hom}_{B\text{-alg}}(A_B^\vee, B)$ is the group of B-module homomorphisms $A_B^\vee \to B$ through which $\mu_B^\vee$ and $\epsilon_B^\vee$ are respectively compatible with m_B and id_B. Taking B-duals, we identify this group with the group of B-module homomorphisms $B \to A_B$ through which $m_B^\vee$ and $\mathrm{id}_B^\vee$ are respectively compatible with μ_B and ϵ_B. Since we have the canonical isomorphism $\mathrm{Hom}_{B\text{-alg}}(B, A_B) \cong A_B^\times$ which sends each $f \in \mathrm{Hom}_{B\text{-alg}}(B, A_B)$ to $f(1)$, we find

$$\mathrm{Hom}_{B\text{-alg}}(A_B^\vee, B) \cong \left\{ u \in A_B^\times : \mu_B(u) = u \otimes u,\ \epsilon_B(u) = 1 \right\}. \tag{1.4}$$

Moreover, the group scheme axioms for G_B yields the relation $(\epsilon_B \otimes \mathrm{id}_B) \circ \mu_B = \mathrm{id}_B$ and consequently implies that every $u \in A_B^\times$ with $\mu_B(u) = u \otimes u$ must satisfy the identity $\epsilon_B(u) = 1$. Hence the isomorphisms (1.3) and (1.4) together yield a natural identification

$$G^\triangledown(B) \cong \left\{ u \in A_B^\times : \mu_B(u) = u \otimes u \right\}. \tag{1.5}$$

Now we establish the desired isomorphism (1.1) by the identifications (1.2) and (1.5), thereby completing the proof of statement (2).

It remains to prove statements (3) and (4). Since $G^\vee$ is commutative by Lemma 1.1.13 and the commutativity of $\mathbb{G}_m$, we deduce statement (3) from statement (2) by observing that $A^\vee$ is locally free of rank n over R. In addition, we apply statements (1) and (2) to see that the canonical R-module isomorphism $A \cong (A^\vee)^\vee$ is indeed an R-algebra isomorphism which respects comultiplications, counits, and coinverses on both sides, thereby establishing statement (4). □

Proposition 1.2.5. Given a finite flat R-group G and an R-algebra B, there exists a natural B-group isomorphism $G^\vee \times_R B \cong (G \times_R B)^\vee$.

Proof. It is evident that $G^\vee \times_R B$ and $(G \times_R B)^\vee$ are naturally isomorphic as group-valued functors. Lemma 1.2.1 and Theorem 1.2.4 together imply that these functors are indeed finite flat B-groups and thus yield the desired assertion. □

Definition 1.2.6. Given a homomorphism $f : G \to H$ of finite flat R-groups, we refer to the induced homomorphism $f^\vee : G^\vee \to H^\vee$ as the *dual homomorphism* of f.

Example 1.2.7. Given a finite flat R-group G, we have $[n]_G^\vee = [n]_{G^\vee}$ for every integer $n > 0$; indeed, $[n]_G^\vee$ maps each $f \in G^\vee(B) = \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B)$ for an arbitrary R-algebra B to $f \circ [n]_{G_B} = [n]_{G^\vee}(f)$.

PROPOSITION 1.2.8. For every positive integer n, we have $(\underline{\mathbb{Z}/n\mathbb{Z}})^\vee \cong \mu_n$ and $\mu_n^\vee \cong \underline{\mathbb{Z}/n\mathbb{Z}}$.

PROOF. Let us set $A := \prod_{i \in \mathbb{Z}/n\mathbb{Z}} R$ and write e_i for the element of A whose only nonzero entry is 1 in the component corresponding to i. As explained in Example 1.1.8 we have $\underline{\mathbb{Z}/n\mathbb{Z}} \cong \operatorname{Spec}(A)$ with comultiplication μ, counit ϵ, and coinverse ι given by the relations

$$\mu(e_i) = \sum_{v+w=i} e_v \otimes e_w, \qquad \epsilon(e_i) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}, \qquad \iota(e_i) = e_{-i}.$$

Let $m_A : A \otimes_R A \to A$ and $s : R \to A$ respectively denote the ring multiplication map and structure morphism of A. We have the dual basis $\{ f_i \}$ of $A^\vee$ with

$$f_i(e_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.2.4 yields a natural identification $(\underline{\mathbb{Z}/n\mathbb{Z}})^\vee \cong \operatorname{Spec}(A^\vee)$ with comultiplication $m_A^\vee$, counit $s^\vee$, and coinverse $\iota^\vee$, where $A^\vee$ is an R-algebra with structure morphism $\epsilon^\vee$ and ring multiplication map $\mu^\vee$. The maps $\mu^\vee$, $\epsilon^\vee$, $m_A^\vee$, $s^\vee$, and $\iota^\vee$ are determined by the identities

$$\mu^\vee(f_i \otimes f_j) = f_{i+j}, \quad \epsilon^\vee(1) = f_0, \quad m_A^\vee(f_i) = f_i \otimes f_i, \quad s^\vee(f_i) = 1, \quad \iota^\vee(f_i) = f_{-i}.$$

Hence the map $A^\vee \to R[t]/(t^n - 1)$ sending each f_i to t^i induces an R-group isomorphism $(\underline{\mathbb{Z}/n\mathbb{Z}})^\vee \cong \mu_n$ by Example 1.1.8 and in turn yields an R-group isomorphism $\mu_n^\vee \cong \underline{\mathbb{Z}/n\mathbb{Z}}$ by Theorem 1.2.4. □

PROPOSITION 1.2.9. If R has characteristic p, the R-group α_p is self-dual.

PROOF. As explained in Example 1.1.8, we have $\alpha_p = \operatorname{Spec}(A)$ for $A := R[t]/(t^p)$ with comultiplication μ, counit ϵ, and coinverse ι given by the relations

$$\mu(t^i) = \sum_{v+w=i} \binom{i}{v} t^v \otimes t^w, \qquad \epsilon(t^i) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}, \qquad \iota(t^i) = (-t)^i.$$

Let $m_A : A \otimes_R A \to A$ and $s : R \to A$ respectively denote the ring multiplication map and structure morphism of A. We have the dual basis $\{ f_i \}$ of $A^\vee$ with

$$f_i(t^j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.2.4 yields a canonical identification $\alpha_p^\vee \cong \operatorname{Spec}(A^\vee)$ with comultiplication $m_A^\vee$, counit $s^\vee$, and coinverse $\iota^\vee$, where $A^\vee$ is an R-algebra with structure morphism $\epsilon^\vee$ and ring multiplication map $\mu^\vee$. The maps $\mu^\vee$, $\epsilon^\vee$, $m_A^\vee$, $s^\vee$, and $\iota^\vee$ are determined by the identities

$$\mu^\vee(f_i \otimes f_j) = \binom{i+j}{i} f_{i+j}, \quad \epsilon^\vee(1) = 0,$$

$$m_A^\vee(f_i) = \sum_{v+w=i} f_v \otimes f_w, \quad s^\vee(f_i) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}, \quad \iota^\vee(f_i) = (-1)^i f_i.$$

Hence the map $A^\vee \to A$ sending each f_i to $t^i/i!$ yields an R-group isomorphism $\alpha_p^\vee \cong \alpha_p$. □

Remark. When R has characteristic p, we have an R-scheme isomorphism $\mu_p \simeq \alpha_p$ given by the ring isomorphism $R[t]/(t^p) \simeq R[t]/(t^p - 1)$ sending t to $t+1$. Propositions 1.2.8 and 1.2.9 together show that μ_p and α_p are not isomorphic as group schemes.

Proposition 1.2.10. Given an abelian scheme $\mathcal{A}$ over R with dual abelian scheme $\mathcal{A}^\vee$, we have a natural isomorphism $\mathcal{A}[n]^\vee \cong \mathcal{A}^\vee[n]$ for every positive integer n.

Proof. The homomorphism $[n]_\mathcal{A}$ is surjective by a standard fact about abelian varieties stated in the Stacks Project [**Sta**, Tag 03RP]. Hence we have a short exact sequence

$$\underline{0} \longrightarrow \mathcal{A}[n] \longrightarrow \mathcal{A} \xrightarrow{[n]} \mathcal{A} \longrightarrow \underline{0}$$

which gives rise to a long exact sequence

$$\underline{0} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{A}, \mathbb{G}_m) \xrightarrow{[n]} \underline{\mathrm{Hom}}(\mathcal{A}, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{A}[n], \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1(\mathcal{A}, \mathbb{G}_m) \xrightarrow{[n]} \underline{\mathrm{Ext}}^1(\mathcal{A}, \mathbb{G}_m).$$

In addition, we have natural identifications

$$\underline{\mathrm{Hom}}(\mathcal{A}, \mathbb{G}_m) = \underline{0}, \qquad \underline{\mathrm{Hom}}(\mathcal{A}[n], \mathbb{G}_m) \cong \mathcal{A}[n]^\vee, \qquad \underline{\mathrm{Ext}}^1(\mathcal{A}, \mathbb{G}_m) \cong \mathcal{A}^\vee$$

by definition of Cartier duals and some general fact about abelian varieties stated in the notes of Milne [**Mil**, §9]. Therefore we obtain an exact sequence

$$\underline{0} \longrightarrow \mathcal{A}[n]^\vee \longrightarrow \mathcal{A}^\vee \xrightarrow{[n]} \mathcal{A}^\vee$$

which yields the desired isomorphism $\mathcal{A}[n]^\vee \cong \mathcal{A}^\vee[n]$. □

Example 1.2.11. If $R = k$ is a field, every elliptic curve E over k admits a natural isomorphism $E[n]^\vee \cong E[n]$ for each integer $n \geq 1$ by Proposition 1.2.10 a standard fact that elliptic curves are self-dual as stated in the notes of Milne [**Mil**, §9].

Lemma 1.2.12. Given a closed embedding $f : H \hookrightarrow G$ of finite flat R-groups, we have a canonical isomorphism $\ker(f^\vee) \cong (G/H)^\vee$.

Proof. Let B be an arbitrary R-algebra and $f_B : H_B \hookrightarrow G_B$ denote the homomorphism induced by f. Theorem 1.1.17 and Lemma 1.2.1 together yield a canonical isomorphism $G_B/H_B \cong (G/H)_B$. Hence we obtain an identification

$$\begin{aligned} \ker(f^\vee)(B) &= \{ g \in \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B) : g \circ f_B = 0 \} \\ &= \{ g \in \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B) : H_B \subseteq \ker(g) \} \\ &\cong \mathrm{Hom}_{B\text{-grp}}(G_B/H_B, (\mathbb{G}_m)_B) \cong \mathrm{Hom}_{B\text{-grp}}((G/H)_B, (\mathbb{G}_m)_B) = (G/H)^\vee(B), \end{aligned}$$

thereby establishing the desired assertion. □

Proposition 1.2.13. Given a short exact sequence of finite flat R-groups

$$\underline{0} \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow \underline{0},$$

the Cartier duality gives rise to a short exact sequence

$$\underline{0} \longrightarrow G''^\vee \longrightarrow G^\vee \longrightarrow G'^\vee \longrightarrow \underline{0}.$$

Proof. Let f and g respectively denote the maps $G' \to G$ and $G \to G''$ in the given short exact sequence. It is straightforward to verify the injectivity of $g^\vee$ by the surjectivity of g. In addition, Lemma 1.2.12 yields a canonical isomorphism $\ker(f^\vee) \cong G''^\vee$. Therefore it remains to establish the surjectivity of $f^\vee$. Since $G''^\vee$ is a finite flat closed R-subgroup of $G^\vee$ by Proposition 1.1.10 and Theorem 1.2.4, we obtain the quotient R-group $G^\vee/G''^\vee$ by Theorem 1.1.17. Now $f^\vee$ factors through a homomorphism $G^\vee/G''^\vee \to G'^\vee$, whose dual coincides with the isomorphism $\ker(g) \cong G'$ induced by f under the identifications

$$(G'^\vee)^\vee \cong G' \qquad \text{and} \qquad (G^\vee/G''^\vee)^\vee \cong \ker((g^\vee)^\vee) \cong \ker(g)$$

given by Theorem 1.2.4 and Lemma 1.2.12. Hence we deduce that $f^\vee$ is surjective as desired, thereby completing the proof. □

1.3. Finite étale group schemes

In this subsection, we introduce finite étale group schemes and discuss their properties.

Definition 1.3.1. Let $G = \operatorname{Spec}(A)$ be an affine R-group. We say that G is *finite étale* if it is finite flat with $\Omega_{A/R} = 0$, where $\Omega_{A/R}$ denotes the module of relative differentials.

LEMMA 1.3.2. Let $G = \operatorname{Spec}(A)$ be a commutative affine R-group.

(1) G is finite étale if and only if its structure morphism to $\operatorname{Spec}(R)$ is finite étale.
(2) When $R = k$ is a field, G is finite étale if and only if there exists a k-algebra isomorphism $A \simeq \prod_{i=1}^{n} k_i$ where each k_i is a finite separable extension of k.

PROOF. The first assertion is an immediate consequencel of Lemma 1.1.13. The second assertion follows from the first assertion by a standard fact about étale morphisms stated in the Stacks project [**Sta**, Tag 00U3]. □

LEMMA 1.3.3. Given a finite étale R-group G and an R-algebra B, the B-scheme G_B is a finite étale B-group.

PROOF. The assertion follows from Lemma 1.2.1, Lemma 1.3.2, and a standard fact that a base change of an étale morphism is étale as stated in the Stacks project [**Sta**, Tag 02GO]. □

PROPOSITION 1.3.4. Assume that R is a henselian local ring with perfect residue field k.

(1) There exists an equivalence of categories
$$\{\text{ finite étale } R\text{-groups }\} \xrightarrow{\sim} \{\text{ finite abelian groups with a continuous } \Gamma_k\text{-action }\}$$
which sends each finite étale R-group G to $G(\overline{k})$.
(2) If a finite étale R-group G has order n, the abelian group $G(\overline{k})$ also has order n.

PROOF. Let us first consider statement (1). By some standard facts about finite étale morphisms stated in the Stacks project [**Sta**, Tag 09ZS and Tag 0BQ8], there exists an equivalence of categories

$$\{\text{ finite étale } R\text{-schemes }\} \xrightarrow{\sim} \{\text{ finite sets with a continuous } \Gamma_k\text{-action }\}$$

which maps each R-scheme T to $T(\overline{k})$. Hence we obtain the desired equivalence by passing to the corresponding categories of commutative group objects.

For statement (2), we write $G = \operatorname{Spec}(A)$ for some locally free R-algebra A of rank n. By Lemma 1.3.2 and Lemma 1.3.3, there exists a k-algebra isomorphism $A \otimes_R k \simeq \prod_{i=1}^{m} k_i$ where each k_i is a finite separable extension of k. Hence we find

$$G(\overline{k}) \cong \operatorname{Hom}_{R\text{-alg}}(A, \overline{k}) \cong \operatorname{Hom}_{R\text{-alg}}(A \otimes_R k, \overline{k}) \simeq \operatorname{Hom}_{R\text{-alg}}\Big(\prod_{i=1}^{m} k_i, \overline{k}\Big) \cong \prod_{i=1}^{m} \operatorname{Hom}_k(k_i, \overline{k})$$

and in turn deduce that the order of $G(\overline{k})$ is

$$\sum_{i=1}^{m} \dim_k(k_i) = \dim_k(A \otimes_R k) = n,$$

thereby completing the proof. □

Remark. Primary examples of henselian local rings are complete local rings and fields.

LEMMA 1.3.5. For an affine R-group $G = \operatorname{Spec}(A)$ with augmentation ideal I, we have a canonical R-module isomorphism $A \cong R \oplus I$.

PROOF. The assertion follows from the observation that the structure morphism $R \to A$ splits the short exact sequence

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\epsilon} R \longrightarrow 0$$

where ϵ denotes the counit of G. □

PROPOSITION 1.3.6. Let $G = \operatorname{Spec}(A)$ be a finite flat R-group with augmentation ideal I.

(1) There exist natural isomorphisms
$$I/I^2 \otimes_R A \cong \Omega_{A/R} \qquad \text{and} \qquad I/I^2 \cong \Omega_{A/R} \otimes_A A/I.$$
(2) G is étale if and only if we have $I = I^2$.

PROOF. Let us consider a commutative diagram

$$\begin{array}{ccc} G \times_R G & \xrightarrow{(g,h)\mapsto(g,gh^{-1})} & G \times_R G \\ & {\scriptstyle\Delta}\nwarrow \quad \nearrow{\scriptstyle(\mathrm{id},e)} & \\ & G & \end{array}$$

where Δ and e respectively denote the diagonal morphism and the unit section of G. The horizontal map is an isomorphism of R-schemes; indeed, it has an inverse which sends each $(g, h) \in G \times_R G$ to $(g, h^{-1}g)$. Hence we obtain a commutative diagram

$$\begin{array}{ccc} A \otimes_R A & \xleftarrow{\ \sim\ } & A \otimes_R A \\ & {\scriptstyle a\otimes b\mapsto ab}\searrow \quad \swarrow{\scriptstyle a\otimes b\mapsto a\cdot\epsilon(b)} & \\ & A & \end{array}$$

where ϵ denotes the counit of G. The horizontal map induces an isomorphism between the kernels of the two downward maps. Let J denote the kernel of the left downward map. Under the canonical decomposition

$$A \otimes_R A \cong A \otimes_R R \oplus A \otimes_R I$$

given by Lemma 1.3.5, we identify the kernel of the right downward map with $A \otimes_R I$ and consequently obtain a natural isomorphism $J \cong A \otimes_R I$. Therefore we have

$$\Omega_{A/R} \cong J/J^2 \cong (A \otimes_R I)/(A \otimes_R I)^2 \cong (A \otimes_R I)/(A \otimes_R I^2) \cong A \otimes_R (I/I^2),$$

where the first identification comes from a standard fact about relative differentials stated in the Stacks project [**Sta**, Tag 00RW], and thus find

$$\Omega_{A/R} \otimes_A (A/I) \cong \big((I/I^2) \otimes_R A\big) \otimes_A A/I \cong (I/I^2) \otimes_R A/I \cong (I/I^2) \otimes_R R \cong I/I^2.$$

Now we see that $\Omega_{A/R}$ vanishes if and only if I/I^2 vanishes, thereby completing the proof. □

Remark. Let us provide some geometric intuition behind the isomorphisms in statement (1). Since $\Omega_{R/R}$ vanishes, we can alternatively obtain the isomorphism $I/I^2 \cong \Omega_{A/R} \otimes_A A/I$ from the conormal exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{A/R} \otimes_A A/I \longrightarrow \Omega_{R/R} \longrightarrow 0$$

given by a standard fact stated in the Stacks project [**Sta**, Tag 06AA]. The isomorphism $I/I^2 \otimes_R A \cong \Omega_{A/R}$ says that we can recover $\Omega_{G/\operatorname{Spec}(R)} = \Omega_{A/R}$ from its pullback along the unit section by multiplying functions on G.

PROPOSITION 1.3.7. Every finite flat constant group scheme is étale.

PROOF. Let M be a finite abelian group with identity element denoted by 0. By the affine description in Example 1.1.8, we have

$$\underline{M} \simeq \mathrm{Spec}\left(\prod_{i\in M} R\right)$$

with counit given by the projection to the factor for $i = 0$. Hence the augment ideal of $\underline{M}$ is

$$I = \prod_{\substack{i\in M\\ i\neq 0}} R.$$

Since I is naturally a multiplicative monoid, we have $I = I^2$. Therefore Proposition 1.3.6 implies that $\underline{M}$ is étale. □

PROPOSITION 1.3.8. Assume that $R = k$ is an algebraically closed field.

(1) Every finite étale k-groups is a constant group scheme.
(2) Given a prime p, the k-group $\underline{\mathbb{Z}/p\mathbb{Z}}$ is a unique finite étale k-group of order p.

PROOF. Proposition 1.3.4 yields an equivalence of categories

$$\{\text{ finite étale } k\text{-groups }\} \xrightarrow{\sim} \{\text{ finite abelian groups }\}$$

which sends each finite étale k-group G to $G(k)$. For every finite abelian group M, we find $\underline{M}(k) \cong M$ by Example 1.1.8. Hence we establish the desired assertions by Proposition 1.3.7 and the fact that $\mathbb{Z}/p\mathbb{Z}$ is a unique group of order p. □

PROPOSITION 1.3.9. A finite flat R-group G is étale if and only if the (scheme theoretic) image of the unit section is open.

PROOF. Let us write $G = \mathrm{Spec}\,(A)$ for some locally free R-algebra A of finite rank and denote by I the augmentation ideal of G. We naturally identify the (scheme theoretic) image of the unit section with $\mathrm{Spec}\,(A/I)$. By Proposition 1.3.6, it suffices to show that the closed embedding $\mathrm{Spec}\,(A/I) \hookrightarrow \mathrm{Spec}\,(A)$ is open if and only if I/I^2 vanishes.

Suppose that I/I^2 vanishes. By Nakayama's lemma, there exists an element $a \in A$ with $a-1 \in I$ and $aI = 0$. We observe that a is idempotent; indeed, we find $a^2 = a(a-1)+a = a$. Let us consider the localization map $A \to A_a$, which is surjective since we have

$$\frac{b}{a^n} = \frac{ba}{a^{n+1}} = \frac{ba}{a} = \frac{b}{1} \qquad \text{for each } b \in A \text{ and } n \geq 1.$$

Its kernel consists of elements $b \in A$ with $a^n b = 0$ for some $n \geq 1$, or equivalently $ab = 0$ as a is idempotent. It contains I since the element a annihilates I, while for every element b in the kernel we have $b = -(a-1)b + ab = -(a-1)b \in I$. Hence the localization map $A \to A_a$ has I as its kernel and thus induces an isomorphism $A/I \cong A_a$. It is now evident that the closed embedding $\mathrm{Spec}\,(A/I) \hookrightarrow \mathrm{Spec}\,(A)$ is open.

For the converse, we now assume that the embedding $\mathrm{Spec}\,(A/I) \hookrightarrow \mathrm{Spec}\,(A)$ is open. Since open embeddings are flat as stated in the Stacks project [**Sta**, Tag 0250], the ring homomorphism $A \twoheadrightarrow A/I$ must be flat. Therefore we obtain a short exact sequence

$$0 \longrightarrow I \otimes_A A/I \longrightarrow A \otimes_A A/I \longrightarrow A/I \otimes_A A/I \longrightarrow 0,$$

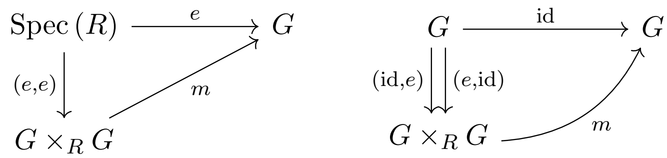
which in turn yields a short exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow A/I \longrightarrow A/I \longrightarrow 0$$

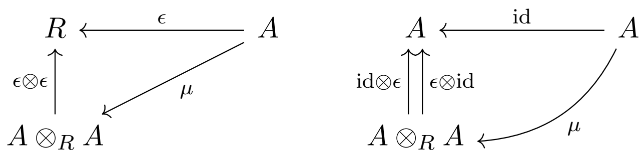
with the third arrow being the identity map. We thus deduce that I/I^2 vanishes as desired. □

THEOREM 1.3.10. A finite flat R-group G with order invertible in R must be étale.

PROOF. Let us write $G = \mathrm{Spec}\,(A)$ for some locally free R-algebra A of finite rank. The group axioms for G yield commutative diagrams



where m and e respectively denote the multiplication map and unit section of G. These diagrams are equivalent to the commutative diagrams



(1.6)

where μ and ϵ respectively denotes the comultiplication and counit of G. Let us denote the augmentation ideal of G by I and take an arbitrary element $t \in I$. We have $\epsilon(t) = 0$ and thus find $\mu(t) \in \ker(\epsilon \otimes \epsilon)$ by the diagram (1.6). Under the decomposition

$$A \otimes_R A \cong (R \otimes_R R) \oplus (I \otimes_R R) \oplus (R \otimes_R I) \oplus (I \otimes_R I)$$

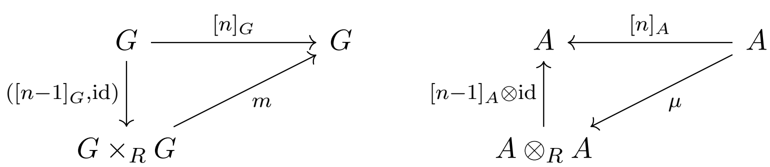
given by Lemma 1.3.5, we obtain a natural identification

$$\ker(\epsilon \otimes \epsilon) \cong (I \otimes_R R) \oplus (R \otimes_R I) \oplus (I \otimes_R I)$$

and thus have $\mu(t) \in a \otimes 1 + 1 \otimes b + I \otimes_R I$ for some $a, b \in I$. Now the diagram (1.6) implies that a and b are both equal to t, thereby yielding the relation

$$\mu(t) \in t \otimes 1 + 1 \otimes t + I \otimes_R I. \tag{1.7}$$

We assert that $[n]_G$ for each $n \geq 1$ induces multiplication by n on I/I^2. Let $[n]_A : A \to A$ denote the R-algebra homomorphism induced by $[n]_G$. We have commutative diagrams



and thus apply the relation (1.7) to find $[n]_A(t) \in [n-1]_A(t) + t + I^2$. Since $[1]_A$ is the identity map on A, we obtain the relation $[n]_A(t) \in nt + I^2$ for each $n \geq 1$ by induction, thereby deducing the desired assertion as t is an arbitrary element in I.

Let us denote the order of G by m. Since $[m]_G$ factors through the unit section of G by Theorem 1.1.16, its induced map on $\Omega_{A/R}$ factors through $\Omega_{R/R} = 0$ and thus must be zero. We find that $[m]_G$ induces a zero map on $I/I^2 \cong \Omega_{A/R} \otimes_A A/I$ by Proposition 1.3.6. Meanwhile, $[m]_G$ induces multiplication by m on I/I^2 and thus is an isomorphism as m is invertible in R. Hence we deduce that I/I^2 vanishes, thereby completing the proof by Proposition 1.3.6. □

Remark. Theorem 1.3.10 is the only result which relies on Theorem 1.1.16 in our discussion. If R is a field, it is possible to prove Theorem 1.3.10 without using Theorem 1.1.16.

COROLLARY 1.3.11. Every finite flat group scheme over a field of characteristic 0 is étale.

1.4. The connected-étale sequence

Throughout this subsection, we assume that R is a henselian local ring and denote its residue field by k. Our main goal for this subsection is to discuss a fundamental theorem that every finite flat R-group naturally arises as an extension of an étale R-group by a connected R-group.

LEMMA 1.4.1. A finite flat R-scheme is étale if and only if its special fiber is étale.

PROOF. The assertion immediately follows from some standard facts about étale morphisms stated in the Stacks project [**Sta**, Tag 02GO, Tag 02GM, and Tag 00U3]. □

Remark. Our proof shows that Lemma 1.4.1 does not require R to be henselian.

LEMMA 1.4.2. For a finite R-scheme T, we have the following equivalent conditions:

(i) T is connected.
(ii) T is a spectrum of a henselian local finite R-algebra.
(iii) The action of Γ_k on $T(\overline{k})$ is transitive.

PROOF. Let us write $T = \mathrm{Spec}\,(B)$ for some finite R-algebra B. By a general fact about henselian local rings stated in the Stacks project [**Sta**, Tag 04GH], we have

$$B \simeq \prod_{i=1}^{n} B_i$$

where each B_i is a henselian local finite R-algebra. Since the spectrum of a local ring is connected, each $T_i := \mathrm{Spec}\,(B_i)$ corresponds to a connected component of T. Hence we deduce the equivalence between conditions (i) and (ii)

We denote the residue field of each B_i by k_i. Via the isomorphism

$$T(\overline{k}) = \mathrm{Hom}_{R\text{-alg}}(B, \overline{k}) \simeq \coprod_{i=1}^{n} \mathrm{Hom}_k(k_i, \overline{k}),$$

we identify each $\mathrm{Hom}_k(k_i, \overline{k})$ as an orbit under the action of Γ_k on $T(\overline{k})$. Therefore we obtain the equivalence between conditions (i) and (iii). □

Remark. If k is algebraically closed, Lemma 1.4.2 shows that a finite R-scheme T is connected if and only if $T(k)$ is a singleton.

LEMMA 1.4.3. A finite R-scheme is connected if and only if its special fiber is connected.

PROOF. The assertion is evident by Lemma 1.4.2. □

Remark. Lemma 1.4.3 is a special case of a general fact that for every proper R-scheme T there exists a natural bijection between the connected components of T and the connected components of T_k, as stated in SGA 4 1/2, Exp. 1, Proposition 4.2.1.

LEMMA 1.4.4. Connected components of a finite flat R-scheme T are finite flat over R.

PROOF. Let T° be a connected component of T. The closed embedding $T^\circ \hookrightarrow T$ is finite flat by general facts stated in the Stacks project [**Sta**, Tag 035C, Tag 04PX]. Hence T° is finite flat over R by a standard fact that the composition of finite flat morphisms is finite flat as stated in the Stacks project [**Sta**, Tag 01WK, Tag 01U7]. □

Remark. Our proof shows that Lemma 1.4.4 holds without any assumption on the base ring.

Definition 1.4.5. Given an R-group G, its *identity component* G° is the connected component of the unit section.

LEMMA 1.4.6. For a finite flat R-group G, we have $G^\circ(\overline{k}) = 0$.

PROOF. Let us write $G = \mathrm{Spec}\,(A)$ for some locally free R-algebra A of finite rank. By Lemma 1.4.2 and Lemma 1.4.4, we have $G^\circ = \mathrm{Spec}\,(A^\circ)$ for some henselian local finite R-algebra A°. Since the unit section factors through G°, it induces a surjective ring homomorphism $A^\circ \to R$. We denote its kernel by I° and obtain an isomorphism $A^\circ/I^\circ \cong R$, which induces an isomorphism between the residue fields of A° and R. Hence we find

$$G^\circ(\overline{k}) = \mathrm{Hom}_{R\text{-alg}}(A^\circ, \overline{k}) \cong \mathrm{Hom}_k(k, \overline{k}) = 0$$

as desired. □

PROPOSITION 1.4.7. A finite flat R-group G is connected if and only if we have $G(\overline{k}) = 0$.

PROOF. If $G(\overline{k})$ is trivial, G is connected by Lemma 1.4.2. Conversely, if G is connected, we have $G = G^\circ$ and thus find $G(\overline{k}) = 0$ by Lemma 1.4.6. □

Example 1.4.8. Let us present some primary examples of connected R-groups.

(1) If k has characteristic p, the R-group μ_{p^v} for each integer $v \geq 1$ is connected by Proposition 1.4.7.
(2) If R has characteristic p, the R-group α_p is connected by Proposition 1.4.7.

THEOREM 1.4.9. Let G be a finite flat R-group. The identity component G° is naturally a finite flat closed R-subgroup of G such that the quotient $G^{\text{ét}} := G/G^\circ$ is étale.

PROOF. Let us first prove that G° is a finite flat closed R-subgroup of G. Since we have $(G^\circ \times_R G^\circ)(\overline{k}) \cong G^\circ(\overline{k}) \times G^\circ(\overline{k}) = 0$ by Lemma 1.4.6, the scheme $G^\circ \times_R G^\circ$ is connected by Lemma 1.4.2. Hence the image of $G^\circ \times_R G^\circ$ under the multiplication map lies in G° for being a connected subscheme of G which contains the unit section. Similarly, the image of G° under the inverse map lies in G°. Therefore G° is an R-subgroup of G, which is evidently closed by construction. Moreover, G° is finite flat by Lemma 1.1.13 and Lemma 1.4.4.

We now consider the finite flat R-group $G^{\text{ét}} = G/G^\circ$ given by Theorem 1.1.17. Its unit section G°/G° has an open image as G° is open in G by the noetherian hypothesis on R. Hence we deduce from Proposition 1.3.9 that $G^{\text{ét}}$ is étale, thereby completing the proof. □

Definition 1.4.10. Given a finite flat R-group G, we refer to the short exact sequence

$$\underline{0} \longrightarrow G^\circ \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow \underline{0}$$

given by Theorem 1.4.9 as the *connected-étale sequence* of G.

Example 1.4.11. Let us describe the connected-étale sequence of μ_n for each integer $n \geq 1$. If k has characteristic 0, Corollary 1.3.11 and Lemma 1.4.1 together imply that μ_n is étale, thereby yielding the connected-étale sequence

$$\underline{0} \longrightarrow \underline{0} \longrightarrow \mu_n \xrightarrow{\text{id}} \mu_n \longrightarrow \underline{0}.$$

Let us henceforth assume that k has characteristic p. We may write $n = p^v m$ for some positive integers v and m such that m is not divisible by p. Then we have a short exact sequence

$$\underline{0} \longrightarrow \mu_{p^v} \longrightarrow \mu_n \xrightarrow{[p^v]} \mu_m \longrightarrow \underline{0}. \tag{1.8}$$

The R-group μ_{p^v} is connected as noted in Example 1.4.8. Moreover, since μ_m has order m by Example 1.1.14, it is étale as easily seen by Theorem 1.3.10 and Lemma 1.4.1. Hence the exact sequence (1.8) is indeed the connected-étale sequence of μ_n.

Proposition 1.4.12. Let G be a finite flat R-group.

(1) The natural surjection $G \twoheadrightarrow G^{\text{ét}}$ induces a canonical isomorphism $G(\overline{k}) \cong G^{\text{ét}}(\overline{k})$.
(2) G is étale if and only if we have $G^{\circ} = \underline{0}$.

Proof. The first statement is evident by Lemma 1.4.6 and Theorem 1.4.9. Since the (scheme theoretic) image of the unit section is closed as noted in Proposition 1.1.10, it is open if and only if it coincides with its connected component G°. Therefore the second statement follows from Proposition 1.3.9. $\square$

Proposition 1.4.13. Let $f : G \to H$ be a homomorphism of finite flat R-groups.

(1) If G is connected, f factors through the embedding $H^{\circ} \hookrightarrow H$.
(2) If H is étale, f factors through the surjection $G \twoheadrightarrow G^{\text{ét}}$.
(3) f naturally induces homomorphisms $f^{\circ} : G^{\circ} \to H^{\circ}$ and $f^{\text{ét}} : G^{\text{ét}} \to H^{\text{ét}}$.

Proof. The first statement is evident since the image of G is a connected R-subgroup of H. The second statement follows from the fact that the image of G° lies in H° by the first statement and thus is trivial by Proposition 1.4.12. The last statement is an immediate consequence of the previous two statements. $\square$

Proposition 1.4.14. Let G, G', and G'' be finite flat R-groups with a short exact sequence

$$\underline{0} \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow \underline{0}.$$

(1) The given exact sequence induces short exact sequences
$$\underline{0} \longrightarrow (G')^{\circ} \longrightarrow G^{\circ} \longrightarrow (G'')^{\circ} \longrightarrow \underline{0},$$
$$\underline{0} \longrightarrow (G')^{\text{ét}} \longrightarrow G^{\text{ét}} \longrightarrow (G'')^{\text{ét}} \longrightarrow \underline{0}.$$
(2) G is connected if and only if both G' and G'' are connected.
(3) G is étale if and only if both G' and G'' are étale.

Proof. Theorem 1.4.9 and Proposition 1.4.13 together yield a commutative diagram

$$\begin{array}{ccccccccc}
 & & \underline{0} & & \underline{0} & & \underline{0} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\underline{0} & \longrightarrow & (G')^{\circ} & \longrightarrow & G' & \longrightarrow & (G')^{\text{ét}} & \longrightarrow & \underline{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\underline{0} & \longrightarrow & G^{\circ} & \longrightarrow & G & \longrightarrow & G^{\text{ét}} & \longrightarrow & \underline{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\underline{0} & \longrightarrow & (G'')^{\circ} & \longrightarrow & G'' & \longrightarrow & (G'')^{\text{ét}} & \longrightarrow & \underline{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \underline{0} & & \underline{0} & & \underline{0} & &
\end{array}$$

where the rows are exact. Since the middle column is exact, Proposition 1.4.12 implies that the right column is exact on the level of $\overline{k}$-points. We deduce from Proposition 1.3.4 that the right column is exact and consequently find by the snake lemma (or the nine lemma) that the left column is exact as well, thereby establishing statement (1). Statement (2) is an immediate consequence of Proposition 1.4.7. Statement (3) follows form the first statement by Proposition 1.4.12. $\square$

Proposition 1.4.15. Assume that $R = k$ is a perfect field. For every finite flat k-group G, the connected-étale sequence canonically splits.

Proof. Let G^{red} denote the reduction of G. If we write $G = \text{Spec}(A)$ for some finite dimensional k-algebra A, we have $G^{\text{red}} = \text{Spec}(A^{\text{red}})$ for $A^{\text{red}} := A/\mathfrak{n}$ where $\mathfrak{n}$ denotes the nilradical of A. We wish to prove that the homomorphism $G \twoheadrightarrow G^{\text{ét}}$ admits a canonical section induced by the closed embedding $G^{\text{red}} \hookrightarrow G$.

We assert that G^{red} is a k-subgroup of G. The scheme $G^{\text{red}} \times_k G^{\text{red}}$ is reduced by a general fact that the product of two reduced schemes over a perfect field is reduced as noted in the Stacks project [**Sta**, Tag 035Z]. Hence the image of $G^{\text{red}} \times_k G^{\text{red}}$ under the multiplication map lies in G^{red} by a standard fact stated in the Stacks project [**Sta**, Tag 0356]. Similarly, the image of G^{red} under the inverse map lies in G^{red}. In addition, the unit section of G factors through G^{red} as k is reduced. Therefore G^{red} is a k-subgroup of G as desired.

Let us now prove that G^{red} is finite étale. By construction, the affine ring A^{red} of G^{red} is a finite dimensional k-algebra. Hence we deduce from some general facts stated in the Stacks project [**Sta**, Tag 00J6 and Tag 00JB] that there exists a k-algebra isomorphism

$$A^{\text{red}} \simeq \prod_{i=1}^{n} A_i^{\text{red}}$$

where each A_i^{red} is a finite dimensional local k-algebra with a unique prime ideal. In fact, since A^{red} is reduced, each A_i^{red} is a finite field extension of k, which is separable as k is perfect. Now Lemma 1.3.2 implies that G^{red} is finite étale as desired.

It remains to show that the homomorphism $G^{\text{red}} \hookrightarrow G \twoheadrightarrow G^{\text{ét}}$ is an isomorphism. The embedding $G^{\text{red}} \hookrightarrow G$ induces an isomorphism $G^{\text{red}}(\overline{k}) \cong G(\overline{k})$ as $\overline{k}$ is reduced. Moreover, the surjection $G \twoheadrightarrow G^{\text{ét}}$ induces an isomorphism $G(\overline{k}) \cong G^{\text{ét}}(\overline{k})$ as noted in Proposition 1.4.12. Therefore the homomorphism $G^{\text{red}} \hookrightarrow G \twoheadrightarrow G^{\text{ét}}$ yields an isomorphism $G^{\text{red}}(\overline{k}) \cong G^{\text{ét}}(\overline{k})$ which is clearly Γ_k-equivariant. Since G^{red} and $G^{\text{ét}}$ are both finite étale, we establish the desired assertion by Proposition 1.3.4. □

Example 1.4.16. We say that an elliptic curve E over $\overline{\mathbb{F}}_p$ is *ordinary* if $E[p](\overline{\mathbb{F}}_p)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. We assert that every ordinary elliptic curve E over $\overline{\mathbb{F}}_p$ yields an isomorphism

$$E[p] \simeq \mu_p \times \underline{\mathbb{Z}/p\mathbb{Z}}.$$

Let us consider the connected-étale sequence

$$\underline{0} \longrightarrow E[p]^{\circ} \longrightarrow E[p] \longrightarrow E[p]^{\text{ét}} \longrightarrow \underline{0}. \tag{1.9}$$

We have $E[p]^{\text{ét}}(\overline{\mathbb{F}}_p) \simeq E[p](\overline{\mathbb{F}}_p) \simeq \mathbb{Z}/p\mathbb{Z}$ by Proposition 1.4.12 and thus find $E[p]^{\text{ét}} \simeq \underline{\mathbb{Z}/p\mathbb{Z}}$ by Proposition 1.3.8. Therefore the exact sequence (1.9) induces a dual exact sequence

$$\underline{0} \longrightarrow (\underline{\mathbb{Z}/p\mathbb{Z}})^{\vee} \longrightarrow E[p]^{\vee} \longrightarrow (E[p]^{\circ})^{\vee} \longrightarrow \underline{0} \tag{1.10}$$

by Proposition 1.2.13, where the second arrow is a closed embedding by Proposition 1.1.10. Now we apply Proposition 1.2.8 and Example 1.2.11 to identify the map $(\underline{\mathbb{Z}/p\mathbb{Z}})^{\vee} \hookrightarrow E[p]^{\vee}$ with a closed embedding $\mu_p \hookrightarrow E[p]$, which in turn gives rise to a closed embedding $\mu_p \hookrightarrow E[p]^{\circ}$ by Proposition 1.4.13 and Example 1.4.8. Moreover, as Example 1.1.14 and Proposition 1.1.15 show that $E[p]^{\text{ét}} \simeq \underline{\mathbb{Z}/p\mathbb{Z}}$ and $E[p]$ respectively have order p and p^2, Theorem 1.1.17 implies that $E[p]^{\circ}$ has order $p^2/p = p$. Since μ_p also has order p by Example 1.1.14, the closed embedding $\mu_p \hookrightarrow E[p]^{\circ}$ is indeed an isomorphism by Theorem 1.1.17. Hence we obtain the desired isomorphism by Proposition 1.4.15.

1.5. The Frobenius morphism

For this subsection, we assume that $R = k$ is a field of characteristic p and write σ for the Frobenius endomorphism of k. Finite flat k-groups admit natural homomorphisms induced by σ. In this subsection, we describe these homomorphisms and explore their applications.

Definition 1.5.1. Let $T = \mathrm{Spec}\,(B)$ be an affine k-scheme and r be a positive integer.

(1) The p^r-*Frobenius twists* of B and T are respectively

$$B^{(p^r)} := B \otimes_{k,\sigma^r} k \qquad \text{and} \qquad T^{(p^r)} := T \times_{k,\sigma^r} k = \mathrm{Spec}\,(B^{(p^r)}),$$

where the factor k in the products has σ^r as structure morphism.

(2) The *relative* p^r-*Frobenius* of B is the k-algebra homomorphism $\varphi_B^{[r]} : B^{(p^r)} \to B$ which maps each $b \otimes c \in B^{(p^r)} = B \otimes_{k_\sigma^r} k$ to $c \cdot b^{p^r} \in B$.

(3) The *relative* p^r-*Frobenius* of T is the morphism $\varphi_T^{[r]} : T \to T^{(p^r)}$ induced by $\varphi_B^{[r]}$.

(4) For $r = 1$, we often refer to $\varphi_B := \varphi_B^{[1]}$ and $\varphi_T := \varphi_T^{[1]}$ as the *Frobenii* of B and T.

Remark. We can similarly define the Frobenius twists and relative Frobenii for all k-schemes.

LEMMA 1.5.2. Let $T = \mathrm{Spec}\,(B)$ be an affine k-scheme and r be a positive integer.

(1) The Frobenius twists satisfy recursive relations

$$B^{(p^{r+1})} = \left(B^{(p^r)}\right)^{(p)} \qquad \text{and} \qquad T^{(p^{r+1})} = \left(T^{(p^r)}\right)^{(p)}.$$

(2) The relative Frobenii satisfy recursive relations

$$\varphi_B^{[r+1]} := \varphi_B^{[r]} \circ \varphi_{B^{(p^r)}} \qquad \text{and} \qquad \varphi_T^{[r+1]} = \varphi_{T^{(p^r)}} \circ \varphi_T^{[r]}.$$

PROOF. The assertions are evident by definition. □

PROPOSITION 1.5.3. Let $T = \mathrm{Spec}\,(B)$ be a k-variety with $B = k[t_1, \cdots, t_n]/(f_1, \cdots, f_m)$ for some polynomials $f_1, \cdots, f_m$ in n variables. Fix a positive integer r.

(1) There exists a canonical k-algebra isomorphism

$$B^{(p^r)} \cong k[t_1, \cdots, t_n]/(f_1^{(p^r)}, \cdots, f_m^{(p^r)})$$

with $f_i^{(p^r)}$ obtained from f_i by raising each coefficient to the p^r-th power.

(2) The homomorphism $\varphi_B^{[r]}$ maps each $t_i \in B^{(p^r)}$ to $t_i^{p^r} \in B$.

(3) For a $\overline{k}$-point on T that represents a common root $(c_1, \cdots, c_n)$ of $f_1, \cdots, f_m$, its image under $\varphi_T^{[r]}$ represents the common root $(c_1^{p^r}, \cdots, c_n^{p^r})$ of $f_1^{(p^r)}, \cdots, f_m^{(p^r)}$.

PROOF. Statement (1) is follows from the fact that under the canonical identification $k[t_1, \cdots, t_n]^{(p^r)} \cong k[t_1, \cdots, t_n]$, the natural map $k[t_1, \cdots, t_n] \to k[t_1, \cdots, t_n]^{(p^r)}$ rasies the coefficients of each polynomial to their p^r-th powers. Statement (2) follows immediately from statement (1). Statement (3) is a straightforward consequence of statement (2). □

PROPOSITION 1.5.4. Given an affine k-scheme $T = \mathrm{Spec}\,(B)$ and a positive integer r, the morphism $\varphi_T^{[r]}$ induces a natural bijection $T(\overline{k}) \cong T^{(p^r)}(\overline{k})$.

PROOF. Let $\mathrm{Frob}_T : T \to T$ denote the morphism induced by the p-th power map on B. Under the natural bijection $T^{(p^r)}(\overline{k}) = T(\overline{k}) \times (\mathrm{Spec}\,(k))\,(\overline{k}) \cong T(\overline{k})$ given by the fact that $(\mathrm{Spec}\,(k))\,(\overline{k})$ is a singleton, $\varphi_T^{[r]}$ maps each $t \in T(\overline{k})$ to $\mathrm{Frob}_T^r(t)$ by construction. Hence we establish the desired assertion by observing that Frob_T^r induces a bijection $T(\overline{k}) \cong T(\overline{k})$. □

Definition 1.5.5. Given a morphism $f : T \to U$ of affine k-schemes and a positive integer r, we refer to the induced morphism $f^{(p^r)} : T^{(p^r)} \to U^{(p^r)}$ as the *p^r-Frobenius twist* of f.

Example 1.5.6. Given an arbitrary affine k-scheme $T = \mathrm{Spec}\,(B)$, we show the equality

$$\left(\varphi_T^{[r]}\right)^{(p^s)} = \varphi_{T^{(p^s)}}^{[r]}$$

for any positive integers r and s. For $r = 1$ and $s = 1$, since we have a commutative diagram

$$\begin{CD} T^{(p)} @>{(\varphi_T)^{(p)}}>> T^{(p^2)} @>>> \mathrm{Spec}\,(k) \\ @VVV @VVV @VV{\sigma}V \\ T @>{\varphi_T}>> T^{(p)} @>>> \mathrm{Spec}\,(k) \end{CD}$$

where each square is cartesian, we find $(\varphi_T)^{(p)} = \varphi_{T^{(p)}}$ by observing that the morphism $T^{(p)} \longrightarrow T \xrightarrow{\varphi_T} T^{(p)}$ given by the left square induces the p-th power map on $B^{(p)}$. For $r = 1$ and $s \geq 2$, we have $(\varphi_T)^{(p^s)} = \left((\varphi_T)^{(p^{s-1})}\right)^{(p)}$ and thus proceed by induction to find $(\varphi_T)^{(p^s)} = \varphi_{T^{(p^s)}}$. Finally, for $r \geq 2$ and $s \geq 2$, we have

$$\left(\varphi_T^{[r]}\right)^{(p^s)} = \left(\varphi_{T^{(p^{r-1})}} \circ \varphi_T^{[r-1]}\right)^{(p^s)} = \left(\varphi_{T^{(p^{r-1})}}\right)^{(p^s)} \circ \left(\varphi_T^{[r-1]}\right)^{(p^s)}$$

by Lemma 1.5.2 and thus proceed by induction to obtain the desired equality.

Lemma 1.5.7. Let T and U be affine k-schemes. Take a positive integer r.

(1) There exists a natural isomorphism $(T \times_k U)^{(p^r)} \cong T^{(p^r)} \times_k U^{(p^r)}$ which canonically identifies $\varphi_{(T \times_k U)}^{[r]}$ with $\varphi_T^{[r]} \times_k \varphi_U^{[r]}$.

(2) Every k-scheme morphism $f : T \to U$ gives rises to a commutative diagram

$$\begin{CD} T @>{\varphi_T^{[r]}}>> T^{(p^r)} \\ @V{f}VV @VV{f^{(p^r)}}V \\ U @>{\varphi_U^{[r]}}>> U^{(p^r)} \end{CD}$$

where all maps are k-scheme morphisms.

Proof. The assertions are straightforward to verify using properties of fiber products. □

Proposition 1.5.8. Let G be an affine k-group and r be a positive integer.

(1) The p^r-Frobenius twist $G^{(p^r)}$ is naturally an affine k-group.

(2) The relative p^r-Frobenius $\varphi_G^{[r]}$ is a k-group homomorphism.

(3) If G is finite flat, $G^{(p^r)}$ is finite flat with a natural isomorphism $\left(G^{(p^r)}\right)^\vee \cong (G^\vee)^{(p^r)}$.

Proof. As we have $G^{(p^r)} = G \times_{k,\sigma^r} k$, statements (1) and (3) are evident by Lemma 1.2.1 and Proposition 1.2.5. Statements (2) is a straightforward consequence of Lemma 1.5.7. □

Lemma 1.5.9. Let $f : G \to H$ be a homomorphism of affine k-groups.

(1) The p^r-Frobenius twist $f^{(p^r)}$ is a k-group homomorphism for each $r \geq 1$.

(2) If f is a closed embedding, $f^{(p^r)}$ is also a closed embedding for each $r \geq 1$.

(3) If f is an isomorphism, $f^{(p^r)}$ is also an isomorphism for each $r \geq 1$.

Proof. The first statement is striaghtforward to verify by Lemma 1.5.7. The remaining statements are evident by the construction of the Frobenius twists via base changes. □

Definition 1.5.10. Let G be a finite flat k-group and r be a positive integer.

(1) We define the p^r-*Verschiebung* to be $\psi_G^{[r]} := \left(\varphi_{G^\vee}^{[r]}\right)^\vee$, regarded as a homomorphism from $G^{(p^r)} \cong \left((G^\vee)^{(p^r)}\right)^\vee$ to $G \cong (G^\vee)^\vee$ under the identifications given by Proposition 1.5.8 and Theorem 1.2.4.

(2) For $r = 1$, we often refer to $\psi_G := \psi_G^{[1]} = \varphi_{G^\vee}^\vee$ as the *Verschiebung* of G.

PROPOSITION 1.5.11. We identify the Frobenius and Verschiebung of $\alpha_p,\ \mu_p,\ \underline{\mathbb{Z}/p\mathbb{Z}}$ as follows:

(1) For α_p, we have $\varphi_{\alpha_p} = 0$ and $\psi_{\alpha_p} = 0$.

(2) For μ_p, we have $\varphi_{\mu_p} = 0$ and $\psi_{\mu_p} = \mathrm{id}_{\mu_p}$.

(3) For $\underline{\mathbb{Z}/p\mathbb{Z}}$, we have $\varphi_{\underline{\mathbb{Z}/p\mathbb{Z}}} = \mathrm{id}_{\underline{\mathbb{Z}/p\mathbb{Z}}}$ and $\psi_{\underline{\mathbb{Z}/p\mathbb{Z}}} = 0$.

PROOF. Let us begin with the Frobenii. We use the affine descriptions in Example 1.1.8. For α_p, we find $\alpha_p^{(p)} \cong \alpha_p$ and $\varphi_{\alpha_p} = 0$ by Proposition 1.5.3. For μ_p, we similarly find $\mu_p^{(p)} \cong \mu_p$ and $\varphi_{\mu_p} = 0$. Let us now consider $\underline{\mathbb{Z}/p\mathbb{Z}}$. We write $A := \prod_{i \in \mathbb{Z}/p\mathbb{Z}} k$ for its affine ring and e_i for the element of A whose only nonzero entry is 1 in the component corresponding to i. We have a natural identification

$$A^{(p)} = \left(\prod_{i\in\mathbb{Z}/p\mathbb{Z}} k\right) \otimes_{k,\sigma} k \cong \prod_{i\in\mathbb{Z}/p\mathbb{Z}} (k \otimes_{k,\sigma} k) \cong \prod_{i\in\mathbb{Z}/p\mathbb{Z}} k = A.$$

Hence for each $a = \sum_{i\in\mathbb{Z}/p\mathbb{Z}} c_i e_i \in A$ with $c_i \in k$ we find

$$\varphi_A(a) = \varphi_A\left(\sum_{i\in\mathbb{Z}/p\mathbb{Z}} c_i e_i\right) = \sum_{i\in\mathbb{Z}/p\mathbb{Z}} \varphi_A(c_i e_i) = \sum_{i\in\mathbb{Z}/p\mathbb{Z}} c_i e_i^p = \sum_{i\in\mathbb{Z}/p\mathbb{Z}} c_i e_i = a,$$

thereby deducing that $\varphi_{\underline{\mathbb{Z}/p\mathbb{Z}}}$ coincides with the identity map. Now that we have the desired identifications of the Frobenii, we deduce the identifications for the Verschiebungs from the results on Cartier duals such as Proposition 1.2.8, and Proposition 1.2.9. □

LEMMA 1.5.12. Given a finite flat k-group G, we have $\psi_G^{[r+1]} = \psi_G^{[r]} \circ \psi_{G^{(p^r)}}$ for each $r \geq 1$.

PROOF. The assertion is evident by Lemma 1.5.2. □

LEMMA 1.5.13. Let G and H be finite flat k-group schemes. Take a positive integer r.

(1) There exists a natural isomorphism $(G \times_k H)^{(p^r)} \cong G^{(p^r)} \times_k H^{(p^r)}$ which canonically identifies $\psi_{(G\times_k H)}^{[r]}$ with $\psi_G^{[r]} \times_k \varphi_H^{[r]}$.

(2) Every homorphism $f : G \to H$ of finite flat k-groups induces commutative diagrams

$$\begin{array}{ccc} G & \xrightarrow{\varphi_G^{[r]}} & G^{(p^r)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{(p^r)}} \\ H & \xrightarrow{\varphi_H^{[r]}} & H^{(p^r)} \end{array} \qquad\qquad \begin{array}{ccc} G & \xleftarrow{\psi_G^{[r]}} & G^{(p^r)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{(p^r)}} \\ H & \xleftarrow{\psi_H^{[r]}} & H^{(p^r)} \end{array}$$

where all maps are k-group homomorphisms.

PROOF. By Lemma 1.2.1, fiber products of finite flat k-groups are finite flat k-groups. Hence the assertions follow from Lemma 1.5.7, Proposition 1.5.8, and Lemma 1.5.9. □

Proposition 1.5.14. Let $G = \mathrm{Spec}\,(A)$ be a finite flat k-group. We denote the symmetric group of order p by $\mathfrak{S}_p$, which acts on $A^{\otimes p}$ by permuting factors of pure tensors.

(1) There exists a k-algebra homomorphism $\gamma : (A^{\otimes p})^{\mathfrak{S}_p} \to A^{(p)}$ with the following properties:
  (i) For each $a \in A$, we have $\gamma(a^{\otimes p}) = a \otimes 1$.
  (ii) For each pure tensor in $A^{\otimes p}$ with unequal factors, the sum of elements in its $\mathfrak{S}_p$-orbit maps to 0 under γ.

(2) The k-algebra homomorphism ψ_A induced by ψ_G fits into a commutative diagram

$$\begin{array}{ccccc} & & \overset{\psi_A}{\curvearrowright} & & \\ A & \longrightarrow & (A^{\otimes p})^{\mathfrak{S}_p} & \xrightarrow{\ \gamma\ } & A^{(p)} \\ & \searrow & \big\downarrow & & \\ & & A^{\otimes p} & & \end{array}$$

with the map $A \to A^{\otimes p}$ induced by the comultiplication of G.

Proof. Let us work with the natural k-algebra isomorphisms

$$A \cong (A^\vee)^\vee, \qquad \left(\mathrm{Sym}^p A^\vee\right)^\vee \cong (A^{\otimes p})^{\mathfrak{S}_p}, \qquad A^{(p)} \cong \left((A^\vee)^{(p)}\right)^\vee,$$

given by Theorem 1.2.4, Proposition 1.5.8, and the fact that $\mathrm{Sym}^p(A^\vee)$ is the k-algebra of $\mathfrak{S}_p$-covariants for $(A^\vee)^{\otimes p}$. Since k has characteristic p, we have $(f_1+f_2)^{\otimes p} = f_1^{\otimes p} + f_2^{\otimes p}$ in $\mathrm{Sym}^p(A^\vee)$ for any $f_1,\ f_2 \in A^\vee$. Therefore there exists a unique k-algebra homomorphism $\theta : (A^\vee)^{(p)} \to \mathrm{Sym}^p A^\vee$ which maps each $f \otimes c \in (A^\vee)^{(p)} = A^\vee \otimes_{k,\sigma} k$ to $c \cdot f^{\otimes p} \in \mathrm{Sym}^p A^\vee$. Let us take γ to be the dual of θ. In addition, we identify each $a \in A$ with its image $\mathfrak{e}_a$ under the isomorphism $A \cong (A^\vee)^\vee$. For each $a \in A$ and $f \otimes c \in (A^\vee)^{(p)} = A^\vee \otimes_{k,\sigma} k$, we have

$$\gamma(a^{\otimes p})(f \otimes c) = (\mathfrak{e}_a)^{\otimes p}(c \cdot f^{\otimes p}) = c \cdot f(a)^p = (\mathfrak{e}_a \otimes 1)(f \otimes c)$$

where the last equality follows from the identity $f(a) \otimes c = 1 \otimes (c \cdot f(a)^p)$ in $A \otimes_{k,\sigma} k$. Moreover, given a pure tensor $\otimes a_i \in A^{\otimes p}$ with unequal factors, we denote its $\mathfrak{S}_p$-stabilizer by S and find

$$\gamma\left(\sum_{\tau \in \mathfrak{S}_p/S} \bigotimes_{i=1}^{p} a_{\tau(i)}\right)(f \otimes c) = \sum_{\tau \in \mathfrak{S}_p/S} \left(\bigotimes_{i=1}^{p} \mathfrak{e}_{a_{\tau(i)}}\right)(c \cdot f^{\otimes p}) = c \sum_{\tau \in \mathfrak{S}_p/S} \prod_{i=1}^{p} f(a_i) = 0$$

for each $f \otimes c \in (A^\vee)^{(p)} = A^\vee \otimes_{k,\sigma} k$, where the last equality follows from the fact that the number of elements in $\mathfrak{S}_p/S$ is divisible by p. Therefore we establish statement (1).

Let us now consider statement (2). By construction, $\varphi_{A^\vee}$ fits into a commutative diagram

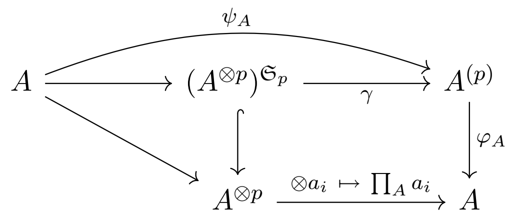
$$\begin{array}{ccccc} & & \overset{\varphi_{A^\vee}}{\curvearrowright} & & \\ (A^\vee)^{(p)} & \xrightarrow{\ \theta\ } & \mathrm{Sym}^p A^\vee & \longrightarrow & A^\vee \\ & & \big\uparrow & \nearrow {\scriptstyle \otimes f_i \mapsto \prod_{A^\vee} f_i} & \\ & & (A^\vee)^{\otimes p} & & \end{array}$$

where $\prod_{A^\vee}$ denotes the ring multiplication on $A^\vee$. Theorem 1.2.4 implies that the dual of the map $(A^\vee)^{\otimes p} \to A^\vee$ in the diagram coincides with the map $A \to A^{\otimes p}$ induced by the comultiplication of G. Since we have $\psi_A = \varphi_{A^\vee}^\vee$ by construction, we obtain the diagram in statement (2) by dualizing the above diagram, thereby completing the proof. $\square$

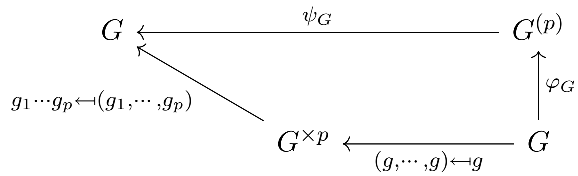
PROPOSITION 1.5.15. Every finite flat k-group G yields the identities

$$\psi_G^{[r]} \circ \varphi_G^{[r]} = [p^r]_G \quad \text{and} \quad \varphi_G^{[r]} \circ \psi_G^{[r]} = [p^r]_{G^{(p)}} \qquad \text{for each integer } r \geq 1.$$

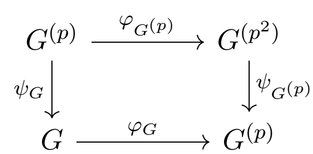
PROOF. An inductive argument based on Lemma 1.5.2 and Lemma 1.5.12 shows that it suffices to establish the desired identities for $r = 1$. Let us write $G = \operatorname{Spec}(A)$ for some finite dimensional k-algebra A. In addition, we let ψ_A denote the k-algebra homomorphism induced by ψ_G and $\mathfrak{S}_p$ denote the symmetric group of order p. Proposition 1.5.14 yields a commutative diagram



with the map $A \to A^{\otimes p}$ induced by the comultiplication of G and $\prod_A$ denoting the ring multiplication on A. Therefore we have a commutative diagram



and in turn find $\psi_G \circ \varphi_G = [p]_G$. Moreover, we have $\varphi_G^{(p)} = \varphi_{G^{(p)}}$ as noted in Example 1.5.6 and thus obtain a commutative diagram



by Lemma 1.5.13. Since we have established the identity $\psi_G \circ \varphi_G = [p]_G$ for an arbitrary finite flat k-group G, we find $\varphi_G \circ \psi_G = \psi_{G^{(p)}} \circ \varphi_{G^{(p)}} = [p]_{G^{(p)}}$ as desired, thereby completing the proof. □

Remark. Let us briefly discuss the Verschiebung for a general affine k-group $G = \operatorname{Spec}(A)$ which is not necessarily finite flat. Our proof of Proposition 1.5.14 readily shows that statement (1) holds for an arbitrary k-algebra A. In addition, the associativity axiom for G implies that the k-algebra homomorphism $A \to A^{\otimes p}$ induced by the comultiplication of G factors through the embedding $(A^{\otimes p})^{\mathfrak{S}_p} \hookrightarrow A^{\otimes p}$. Therefore there exists a unique k-algebra homomorphism $\psi_A : A \to A^{(p)}$ which fits into the diagram in statement (2). We define the Verschiebung of G to be the k-scheme morphism $\psi_G : G^{(p)} \to G$ induced by ψ_A. It is not hard to verify that ψ_A is compatible with comultiplications, which means that ψ_G is a k-group homomorphism. Moreover, for each integer $r \geq 1$ we inductively define the k-group homomorphism $\psi_G^{[r]}$ by the recursive relation in Lemma 1.5.12. It turns out that Lemma 1.5.13 and Proposition 1.5.15 hold for general affine k-groups; indeed, we can establish Lemma 1.5.13 by a straightforward argument on affine rings and in turn deduce Proposition 1.5.15 by the same proof. In addition, we can suitably adjust our argument in Example 1.5.6 to obtain the identity $(\psi_G^{[r]})^{(p^s)} = \psi_{G^{(p^s)}}^{[r]}$ for any positive integers r and s.

LEMMA 1.5.16. Let $G = \operatorname{Spec}(A)$ be a finite flat k-group.

(1) The Frobenius φ_G is an isomorphism if and only if it is injective.
(2) If G is connected, A is an artinian local k-algebra with its maximal ideal given by the augmentation ideal of G.

PROOF. Since G and $G^{(p)}$ are of the same order by construction, statement (1) follows from Proposition 1.1.10 and Theorem 1.1.17. If G is connected, A is an artinian local ring by Lemma 1.1.13, Lemma 1.4.2, and a general fact that every finite dimensional algebra over a field is artinian as noted in the Stacks project [**Sta**, Tag 00J6]. Hence we deduce statement (2) by observing that the augmentation ideal I of G is a maximal ideal as we have $A/I \cong k$. $\square$

PROPOSITION 1.5.17. Let $G = \operatorname{Spec}(A)$ be a finite flat k-group.

(1) G is connected if and only if $\varphi_G^{[r]}$ vanishes for some integer $r \geq 1$.
(2) G is étale if and only if φ_G is an isomorphism.

PROOF. Let us begin with statement (1). If $\varphi_G^{[r]}$ vanishes for some $r \geq 1$, we find by Proposition 1.5.4 that $G(\overline{k})$ is trivial and thus deduce from Proposition 1.4.7 that G is connected. For the converse, we now assume that G is connected. Its augmentation ideal I is nilpotent by Lemma 1.5.16 and a standard fact stated in the Stacks project [**Sta**, Tag 00J8]; in particular, there exists an integer $r \geq 1$ with $t^{p^r} = 0$ for all $t \in I$. Therefore $\varphi_A^{[r]}$ factors through the surjection $A^{(p^r)} = A \otimes_{k,\sigma^r} k \twoheadrightarrow (A/I) \otimes_{k,\sigma^r} k$ induced by the unit section of $G^{(p^r)}$. We deduce that $\varphi_G^{[r]}$ vanishes and in turn establish statement (1).

It remains to prove statement (2). Let us assume that φ_G is an isomorphism. It is not hard to see that φ_{G° is an isomorphism, for example by Lemma 1.5.7 and Lemma 1.5.16. Hence Example 1.5.6 and Lemma 1.5.9 together imply that $\varphi_{(G^\circ)^{(p^r)}} = \varphi_{G^\circ}^{(p^r)}$ is an isomorphism for each $r \geq 1$. Now a simple induction based on Lemma 1.5.2 shows that $\varphi_{G^\circ}^{[r]}$ is an isomorphism for each $r \geq 1$. Since $\varphi_{G^\circ}^{[r]}$ vanishes for some $r \geq 1$ by statement (1), we find that G° is trivial and consequently deduce from Proposition 1.4.12 that G is étale.

We now assume for the converse that G is étale. Since Lemma 1.5.7 implies that $\varphi_{\ker(\varphi_G)}$ vanishes, $\ker(\varphi_G)$ is connected by statement (1); in particular, $\ker(\varphi_G)$ lies in G°. Hence we find by Proposition 1.4.12 that $\ker(\varphi_G)$ is trivial and in turn deduce from Lemma 1.5.16 that φ_G is an isomorphism, thereby completing the proof. $\square$

Remark. Proposition 1.5.17 yields similar criteria for $G^\vee$ to be connected or étale in terms of the Verschiebungs.

Example 1.5.18. Let E be an ordinary elliptic curve over $\overline{\mathbb{F}}_p$. We assert that there exists an isomorphism $\ker(\varphi_{E[p]}) \simeq \mu_p$. Example 1.4.16 shows that we have $E[p]^\circ \simeq \mu_p$. Lemma 1.5.7 and Proposition 1.5.17 together imply that $\ker(\varphi_{E[p]})$ is connected and thus lies in $E[p]^\circ \simeq \mu_p$. On the other hand, $\ker(\varphi_{E[p]})$ contains $E[p]^\circ \simeq \mu_p$ as φ_{μ_p} vanishes by Proposition 1.5.11. Therefore we have $\ker(\varphi_{E[p]}) = E[p]^\circ \simeq \mu_p$ as desired.

Remark. As noted after Definition 1.5.1, we can define the relative Frobenii for general k-schemes, including abelian k-varieties. Moreover, since abelian varieties admit a notion of duality, we can define their relative Verschiebungs as in Definition 1.5.10. It turns out that most results that in this subsection remain valid for abelian varieties. In particular, for an ordinary elliptic curve E over $\overline{\mathbb{F}}_p$, we find $\ker(\varphi_E) \subseteq E[p]$ by the identity $\psi_E^{[r]} \circ \varphi_E^{[r]} = [p^r]_E$ and in turn obtain an isomorphsim $\ker(\varphi_E) \simeq \mu_p$ from Example 1.5.18.

PROPOSITION 1.5.19. Let $G = \mathrm{Spec}\,(A)$ be a finite flat k-group with augmentation ideal I.

(1) For each integer $r \geq 1$, there exists a natural isomorphism

$$\ker(\varphi_G^{[r]}) \cong \mathrm{Spec}\,(A/I^{(p^r)})$$

where $I^{(p^r)}$ denotes the ideal generated by the p^r-th powers of elements in I.

(2) If φ_G vanishes, the order of G is p^d where d denotes the dimension of I/I^2 over k.

PROOF. Let us denote by e the unit section of G, which we naturally identify with the closed embedding $\mathrm{Spec}\,(A/I) \hookrightarrow \mathrm{Spec}\,(A)$. The unit section of $G^{(p^r)}$ is $e^{(p^r)}$, induced by natural surjection $A^{(p^r)} = A \otimes_{k,\sigma^r} k \twoheadrightarrow (A/I) \otimes_{k,\sigma^r} k$. Hence statement (1) follows from the identification of $\ker(\varphi_G^{[r]})$ as the fiber of $\varphi_G^{[r]}$ over $e^{(p^r)}$.

Let us now consider statement (2). We chooose $a_1, \cdots, a_d \in I$ whose images in I/I^2 form a basis over k. Since Proposition 1.5.17 shows that G is connected, we note by Lemma 1.5.16 that A is a local ring with maximal ideal I and in turn deduce from Nakayama's lemma that $a_1, \cdots, a_d$ generate I. Therefore statement (1) yields an isomorphism $A \cong A/(a_1^p, \cdots, a_d^p)$. Let us take the k-algebra homomorphism

$$\lambda : k[t_1, \cdots, t_d] \longrightarrow A \cong A/(a_1^p, \cdots, a_d^p)$$

which maps each t_i to a_i. Since λ is surjective as easily seen by Lemma 1.3.5, we have

$$k[t_1, \cdots, t_d]/\ker(\lambda) \simeq A$$

and thus obtain an isomorphism

$$\Omega_{A/k} \simeq \bigoplus_{i=1}^{d} A \cdot dt_i \Big/ \sum_{f \in \ker(\lambda)} A \cdot df$$

by a general fact about differentials stated in the Stacks project [**Sta**, Tag 00RU]. Moreover, Proposition 1.3.6 implies that $\Omega_{A/k}$ is a free A-module of rank d. Hence we deduce that $\sum_{f \in \ker(\lambda)} A \cdot df$ is trivial, which means that $\ker(\lambda)$ is stable under partial derivatives. Now we must have $\ker(\lambda) \subseteq (t_1^p, \cdots, t_d^p)$, since otherwise we take an element $f \in \ker(\lambda) \setminus (t_1^p, \cdots, t_d^p)$ with minimal sum of degrees of its terms and find that its partial derivatives yield elements in $\ker(\lambda)$ which violate the minimality for f. As $\ker(\lambda)$ evidently contains $(t_1^p, \cdots, t_d^p)$, we obtain an isomorphism $k[t_1, \cdots, t_d]/(t_1^p, \cdots, t_d^p) \simeq A$ and thus establish statement (2) by observing that $k[t_1, \cdots, t_d]/(t_1^p, \cdots, t_d^p)$ is free of dimension p^d over k. $\square$

PROPOSITION 1.5.20. If a finite flat k-group G is connected, its order is a power of p.

PROOF. Let us denote the order of G by n. Since the assertion is trivial for $n = 1$, we henceforth assume $n > 1$ and proceed by induction on n. It is evident by Proposition 1.4.12 that G is not étale. Hence Lemma 1.5.16 and Proposition 1.5.17 together imply that $\ker(\varphi_G)$ is not trivial. In addition, as $\ker(\varphi_G)$ is a closed k-subgroup of G by Proposition 1.1.10, we apply Proposition 1.4.14 to see that both $\ker(\varphi_G)$ and $G/\ker(\varphi_G)$ are connected. Let us write n_1 and n_2 respectively for the orders of $\ker(\varphi_G)$ and $G/\ker(\varphi_G)$. By Theorem 1.1.17 we have $n = n_1 n_2$. If φ_G does not vanish, we find that both n_1 and n_2 are less than n and thus are powers of p by the induction hypothesis, which in particular implies that n is a power of p. If φ_G vanishes, Proposition 1.5.19 readily shows that n is a power of p. Hence we establish the desired assertion. $\square$

Proposition 1.5.21. Given a finite flat k-group $G = \operatorname{Spec}(A)$ with unit section e, its tangent space at e admits a canonical isomorphism $t_{G,e} \cong \operatorname{Hom}_{k\text{-grp}}(G^\vee, \mathbb{G}_a)$.

Proof. Let us write I for the augmentation ideal of G and regard the unit section e as a k-point of G via the natural closed embedding $\operatorname{Spec}(k) \cong \operatorname{Spec}(A/I) \hookrightarrow \operatorname{Spec}(A)$. The tangent space $t_{G,e}$ is by definition canonically isomorphic to the kernel of the natural homomorphism $G(k[t]/(t^2)) \to G(k)$, which we naturally identify with the group of k-algebra homomorphisms $A \to k[t]/(t^2)$ whose composition with the map $k[t]/(t^2) \to k$ equals the counit ϵ of G. Since we can uniquely write every k-linear map $A \to k[t]/(t^2)$ in the form $f_0 + tf_1$ with $f_0,\ f_1 \in A^\vee = \operatorname{Hom}_{k\text{-mod}}(A, k)$, we find

$$\begin{aligned} t_{G,e} &\cong \left\{ f \in \operatorname{Hom}_{k\text{-alg}}(A, k[t]/(t^2)) : f = \epsilon + tg \text{ with } g \in A^\vee \right\} \\ &\cong \left\{ g \in A^\vee : \epsilon + tg \in \operatorname{Hom}_{k\text{-alg}}(A, k[t]/(t^2)) \right\}. \end{aligned}$$

For each $g \in A^\vee$, we have $\epsilon + tg \in \operatorname{Hom}_{k\text{-alg}}(A, k[t]/(t^2))$ if and only if it satisfies the identities

$$\epsilon(ab) + tg(ab) = (\epsilon(a) + tg(a))\,(\epsilon(b) + tg(b)) \quad \text{and} \quad \epsilon(1) + tg(1) = 1 \quad \text{for each } a,\ b \in A,$$

which are equivalent to the identities

$$g(ab) = \epsilon(a)g(b) + \epsilon(b)g(a) \quad \text{and} \quad g(1) = 0 \quad \text{for each } a,\ b \in A$$

by the fact that ϵ is an k-algebra homomorphism. We observe that the second identity is redundant as it follows from the first identity for $a = b = 1$. In addition, the first identity is equivalent to the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad g \quad} & k \cong k \otimes_k k \\ {\scriptstyle m_A}\uparrow & \nearrow_{\epsilon\otimes g + g\otimes\epsilon} & \\ A \otimes_k A & & \end{array}$$

where m_A denotes the ring multiplication map on A. We dualize this diagram under the identification $A^\vee = \operatorname{Hom}_{k\text{-mod}}(A, k) \cong \operatorname{Hom}_{k\text{-mod}}(k, A^\vee)$ and find $m_A^\vee(g) = g \otimes 1 + 1 \otimes g$. Therefore we obtain a natural isomorphsim

$$t_{G,e} \cong \left\{ g \in A^\vee : m_A^\vee(g) = g \otimes 1 + 1 \otimes g \right\}.$$

Meanwhile, by Example 1.1.8 and Theorem 1.2.4 we find

$$\operatorname{Hom}_{k\text{-grp}}(G^\vee, \mathbb{G}_a) \cong \left\{ f \in \operatorname{Hom}_{k\text{-alg}}(k[t], A^\vee) : m_A^\vee(f(t)) = f(t) \otimes 1 + 1 \otimes f(t) \right\}$$

where the identity $m_A^\vee(f(t)) = f(t) \otimes 1 + 1 \otimes f(t)$ comes from compatibility with comultiplications. Since we have the canonical isomorphism $\operatorname{Hom}_{k\text{-alg}}(k[t], A^\vee) \cong A^\vee$ which sends each $f \in \operatorname{Hom}_{k\text{-alg}}(k[t], A^\vee)$ to $f(t)$, we obtain a natural identification

$$\operatorname{Hom}_{k\text{-grp}}(G^\vee, \mathbb{G}_a) \cong \left\{ g \in A^\vee : m_A^\vee(g) = g \otimes 1 + 1 \otimes g \right\}.$$

Therefore we deduce the desired assertion, thereby completing the proof. □

Proposition 1.5.22. A finite flat k-group G is étale if and only if $\operatorname{Hom}_{k\text{-grp}}(G^\vee, \mathbb{G}_a)$ vanishes.

Proof. Let us write $G = \operatorname{Spec}(A)$ for some finite dimensional k-algebra A. We denote the augmentation ideal of G by I and regard the unit section e as a k-point of G via the closed embedding $\operatorname{Spec}(k) \cong \operatorname{Spec}(A/I) \hookrightarrow \operatorname{Spec}(A)$. The tangent space $t_{G,e}$ is naturally isomorphic to the dual of I/I^2 by a general fact stated in the Stacks project [**Sta**, Tag 0B2E]. Therefore, by Proposition 1.3.6, G is étale if and only if $t_{G,e}$ vanishes. Now the desired assertion follows from Proposition 1.5.21. □

THEOREM 1.5.23. Assume that k is algebraically closed.

(1) Every simple finite flat k-group is either étale or connected.
(2) The simple finite étale k-groups are $\underline{\mathbb{Z}/\ell\mathbb{Z}}$ where ℓ ranges over all prime numbers.
(3) The simple connected finite flat k-groups are μ_p and α_p.

PROOF. Statement (1) is straightforward to verify by Theorem 1.4.9. Statement (2) follows from Proposition 1.3.7, Proposition 1.3.8, and the fact that the simple abelian groups are precisely the cyclic groups of prime order. Hence it remains to prove statement (3).

The k-groups μ_p and α_p are indeed connected as noted in Example 1.4.8. Moreover, they are of order p by construction and thus are simple by Theorem 1.4.9. We wish to show that they are the only simple connected finite flat k-groups.

Let G be a simple connected finite flat k-group. Theorem 1.2.4 and Proposition 1.2.13 together imply that $G^\vee$ is simple. Hence $G^\vee$ is either étale or connected by statement (1).

We consider the case where $G^\vee$ is étale. Statement (2) yields an isomorphism $G^\vee \simeq \underline{\mathbb{Z}/\ell\mathbb{Z}}$ for some prime ℓ. Hence G has order ℓ by Example 1.1.14 and Theorem 1.2.4. On the other hand, the order of G is a power of p as noted in Proposition 1.5.20. We thus find $\ell = p$ and in turn obtain an isomorphism $G \simeq \mu_p$ by Proposition 1.2.8.

Let us now consider the case where $G^\vee$ is connected. It is evident by Proposition 1.4.12 that neither G nor $G^\vee$ is étale. Theorem 1.2.4 and Proposition 1.5.22 together yield a nonzero k-group homomorphism $f : G \to \mathbb{G}_a$, which is indeed a closed embedding as $G^\vee$ is simple. Moreover, Lemma 1.5.16 and Proposition 1.5.17 together imply that $\ker(\varphi_G)$ is not trivial, which means that φ_G vanishes as G is simple. Therefore f must factor through $\ker(\varphi_{\mathbb{G}_a})$, which is isomorphic to α_p as easily seen by Example 1.1.8 and Proposition 1.5.3. Since α_p is simple, we deduce that f induces an isomorphism $G \simeq \alpha_p$. □

Remark. In the category of finite flat group schemes, the image of a homomorphism is a scheme theoretic image and thus is closed in the target; in particular, subobjects of a finite flat k-group scheme is a closed k-subgroup.

Example 1.5.24. We say that an elliptic curve E over $\overline{\mathbb{F}}_p$ is *supersingular* if $E[p](\overline{\mathbb{F}}_p)$ is trivial. We assert that every supersingular elliptic curve E over $\overline{\mathbb{F}}_p$ yields a short exact sequence

$$\underline{0} \longrightarrow \alpha_p \longrightarrow E[p] \longrightarrow \alpha_p \longrightarrow \underline{0}.$$

Example 1.1.14 and Theorem 1.5.23 together show that the order of every simple finite flat $\overline{\mathbb{F}}_p$-group is a prime. Since $E[p]$ has order p^2 as noted in Proposition 1.1.15, it is not simple and thus admits a nonzero proper closed $\overline{\mathbb{F}}_p$-subgroup H. Let us consider the exact sequence

$$\underline{0} \longrightarrow H \longrightarrow E[p] \longrightarrow E[p]/H \longrightarrow \underline{0}.$$

Proposition 1.2.13 and Example 1.2.11 together yield a short exact sequence

$$\underline{0} \longrightarrow (E[p]/H)^\vee \longrightarrow E[p] \longrightarrow H^\vee \longrightarrow \underline{0}.$$

Since $E[p]$ is connected as easily seen by Proposition 1.4.7, we deduce from Proposition 1.4.14 that H, $E[p]/H$, $H^\vee$, $(E[p]/H)^\vee$ are all connected. In addition, we find by Theorem 1.1.17 that both H and $E[p]/H$ have order p and thus are simple. Therefore Proposition 1.2.8 and Theorem 1.5.23 together imply that both H and $E[p]/H$ are isomorphic to α_p, thereby yielding the desired assertion.

Remark. It turns out that the $\overline{\mathbb{F}}_p$-subgroup $H \simeq \alpha_p$ coincides with $\ker(\varphi_{E[p]})$.

2. p-divisible groups

In this section, we introduce p-divisible groups as limits of finite flat group schemes and discuss some fundamental theorems about their structures. The primary references for this section are the book of Demazure [**Dem72**] and the article of Tate [**Tat67**]. Throughout this section, we let R denote a noetherian base ring.

2.1. Basic definitions and properties

In this subsection, we define p-divisible groups and describe their basic properties inherited from properties of finite flat group schemes.

Definition 2.1.1. A *p-divisible group of height h* over R is an ind-scheme $G = \varinjlim_{v>0} G_v$ with the following properties:

(i) Each G_v is a finite flat R-group of order p^{vh}.
(ii) Each transition map $i_v : G_v \to G_{v+1}$ fits into a short exact sequence

$$\underline{0} \longrightarrow G_v \xrightarrow{\ i_v\ } G_{v+1} \xrightarrow{\ [p^v]\ } G_{v+1}.$$

Remark. Some authors prefer to say *Barsotti-Tate groups* for p-divisible groups.

Example 2.1.2. We present some important examples of p-divisible groups.

(1) The trivial R-group $\underline{0}$ is a unique p-divisible group of height 0 over R via the identification $\underline{0} \cong \varinjlim \underline{0}$.
(2) The *constant p-divisible group* over R is $\underline{\mathbb{Q}_p/\mathbb{Z}_p} := \varinjlim \underline{\mathbb{Z}/p^v\mathbb{Z}}$ with natural inclusions. It is a p-divisible group of height 1 over R.
(3) The *p-power roots of unity* over R is $\mu_{p^\infty} := \varinjlim \mu_{p^v}$ with natural inclusions. It is a p-divisible group of height 1 over R.
(4) Every abelian scheme $\mathcal{A}$ of dimension g over R gives rises to a p-divisible group $\mathcal{A}[p^\infty] := \varinjlim \mathcal{A}[p^v]$ of height $2g$ over R by Proposition 1.1.15.

Remark. When R has characteristic p, we have a finite flat R-group $\alpha_{p^v} := \mathrm{Spec}\,(R[t]/t^{p^v})$ for each integer $v \geq 1$ with the natural additive group structure on $\alpha_{p^v}(B) = \left\{ b \in B : b^{p^v} = 0 \right\}$ for each R-algebra B. However, the ind-scheme $\varinjlim \alpha_{p^v}$ over R with natural inclusions is not a p-divisible group since $[p]_{\alpha^v}$ vanishes for each $v \geq 1$.

Definition 2.1.3. Let $G = \varinjlim G_v$ and $H = \varinjlim H_v$ be p-divisible groups over R.

(1) A *homomorphism* from G to H is a system $f = (f_v)$ of R-group homomorphisms $f_v : G_v \to H_v$ which fit into commutative diagrams

$$\begin{array}{ccc} G_v & \xrightarrow{\ f_v\ } & H_v \\ {\scriptstyle i_v}\big\downarrow & & \big\downarrow{\scriptstyle j_v} \\ G_{v+1} & \xrightarrow{\ f_{v+1}\ } & H_{v+1} \end{array}$$

where i_v and j_v respectively denote transition maps of G and H.
(2) The *kernel* of a homomorphism $f = (f_v)$ from G to H is $\ker(f) := \varinjlim \ker(f_v)$.

Example 2.1.4. Given a p-divisible group $G = \varinjlim G_v$ over R and an integer n, the *multiplication by n* on G is the homomorphism $[n]_G := ([n]_{G_v})$.

Lemma 2.1.5. Let B be an R-algebra.

(1) Given a p-divisible group $G = \varinjlim G_v$ of height h over R, the base change to B yields a p-divisible group $G_B = \varinjlim (G_v)_B$ of height h over B.

(2) Given a short exact sequence of p-divisible groups over R

$$\underline{0} \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow \underline{0},$$

the base change to B yields a short exact sequence of p-divisible groups

$$\underline{0} \longrightarrow (G')_B \longrightarrow G_B \longrightarrow (G'')_B \longrightarrow \underline{0}.$$

Proof. The assertions are straightforward to verify by Lemma 1.2.1. □

Lemma 2.1.6. Every p-divisible group $G = \varinjlim G_v$ over R yields R-group homomorphisms $i_{v,w} : G_v \to G_{v+w}$ and $j_{v,w} : G_{v+w} \to G_w$ for each $v,\ w \geq 1$ with the following properties:

(i) The map $i_{v,w}$ induces a canonical isomorphism $G_v \cong G_{v+w}[p^v]$.

(ii) There exists a commutative diagram

$$\begin{array}{ccc} G_{v+w} & \xrightarrow{[p^v]} & G_{v+w} \\ & {\scriptstyle j_{v,w}} \searrow \quad \nearrow {\scriptstyle i_{w,v}} & \\ & G_w & \end{array}$$

(iii) We have a short exact sequence

$$\underline{0} \longrightarrow G_v \xrightarrow{i_{v,w}} G_{v+w} \xrightarrow{j_{v,w}} G_w \longrightarrow \underline{0}.$$

Proof. Let us write $i_v : G_v \to G_{v+1}$ for the transition map. For each $v,\ w \geq 1$ the map i_{v+w-1} induces a natural isomorphism

$$G_{v+w}[p^v] \cong G_{v+w}[p^{v+w-1}] \cap G_{v+w}[p^v] \cong G_{v+w-1} \cap G_{v+w}[p^v] \cong G_{v+w-1}[p^v].$$

Hence we set $i_{v,w} := i_{v+w-1} \circ \cdots \circ i_v$ and establish property (i) by induction on w. Moreover, as the image of $[p^v]_{G_{v+w}}$ lies in $G_{v+w}[p^w]$ by the fact that $[p^{v+w}]_{G_{v+w}}$ vanishes, property (i) implies that there exists a unique map $j_{v,w} : G_{v+w} \to G_w$ which satisfies property (ii).

It remains to verify property (iii). The map $i_{v,w}$ is a closed embedding as easily seen by Proposition 1.1.10. In addition properties (i) and (ii) together yield an identification $\ker(j_{v,w}) = G_{v+w}[p^v] \cong G_v$. Hence $j_{v,w}$ gives rise to a closed embedding $G_{v+w}/G_v \hookrightarrow G_w$, which is indeed an isomorphism by Theorem 1.1.17 as both G_{v+w}/G_v and G_w have order p^w. We deduce that $j_{v,w}$ is surjective and consequently establish property (iii). □

Proposition 2.1.7. Let $G = \varinjlim G_v$ be a p-divisible group over R.

(1) There exists a canonical identification $G_v \cong \ker([p^v]_G)$ for each $v \geq 1$.

(2) The homomorphism $[p]_G$ is surjective.

Proof. Given an integer $v \geq 1$, we obtain a natural isomorphism $\ker([p^v]_{G_w}) \cong G_v$ for each $w \geq v$ by Lemma 2.1.6 and thus establish statement (1). In addition, we deduce from Lemma 2.1.6 that the map $[p]_{G_{v+1}}$ factors through a surjection $G_{v+1} \twoheadrightarrow G_v$ for each $v \geq 1$ and consequently establish statement (2). □

Remark. Statement (1) shows that the kernel of a homomorphism between two p-divisible groups is not necessarily a p-divisible group. For statement (2), we may define the surjectivity of $[p]_G$ in terms of fpqc sheaves over R.

Proposition 2.1.8. Let $G = \varinjlim G_v$ be a p-divisible group of height h over R.

(1) The ind-scheme $G^\vee := \varinjlim G_v^\vee$ with transition maps induced by $[p]_G$ is a p-divisible group of height h over R.
(2) There exists a canonical isomorphism $G \cong (G^\vee)^\vee$.

Proof. Lemma 2.1.6 yields a commutative diagram

$$\begin{array}{ccccccccccc} & & & & & G_1 & & & & & \\ & & & & {\scriptstyle j_{v,1}} \nearrow & & \searrow {\scriptstyle i_{1,v}} & & & & \\ \underline{0} & \longrightarrow & G_v & \xrightarrow{i_v = i_{v,1}} & G_{v+1} & \xrightarrow{\quad [p^v] \quad} & & G_{v+1} & \xrightarrow{j_v = j_{1,v}} & G_v & \longrightarrow \underline{0} \end{array}$$

where the horizontal arrows form an exact sequence. Hence we obtain an exact sequence

$$\underline{0} \longrightarrow G_v^\vee \xrightarrow{j_v^\vee} G_{v+1}^\vee \xrightarrow{[p^v]} G_{v+1}^\vee$$

by Example 1.2.7 and Proposition 1.2.13. Now the desired assertions immediately follow from Theorem 1.2.4. □

Definition 2.1.9. Given a p-divisible group G over R, we refer to the p-divisible group $G^\vee$ in Proposition 2.1.8 as the *Cartier dual* of G.

Remark. Some authors prefer to call $G^\vee$ the *Serre dual* of G.

Example 2.1.10. Let us record the Cartier duals of p-divisible groups from Example 2.1.2.

(1) The Cartier dual of $\underline{0}$ is evidently $\underline{0}$ by definition.
(2) We have $(\underline{\mathbb{Q}_p/\mathbb{Z}_p})^\vee \cong \mu_{p^\infty}$ and $\mu_{p^\infty}^\vee \cong \underline{\mathbb{Q}_p/\mathbb{Z}_p}$ by Proposition 1.2.8.
(3) Given an abelian scheme $\mathcal{A}$ over R, we have $\mathcal{A}[p^\infty]^\vee \cong \mathcal{A}^\vee[p^\infty]$ by Proposition 1.2.10 where $\mathcal{A}^\vee$ denotes the dual abelian scheme of $\mathcal{A}$.

Proposition 2.1.11. Assume that R is a henselian local ring with residue field k. Let $G = \varinjlim G_v$ be a p-divisible group over R.

(1) There exists a natural exact sequence of p-divisible groups
$$\underline{0} \longrightarrow G^\circ \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow \underline{0} \tag{2.1}$$
with $G^\circ = \varinjlim G_v^\circ$ and $G^{\text{ét}} = \varinjlim G_v^{\text{ét}}$.
(2) If $R = k$ is a perfect field, the exact sequence (2.1) canonically splits.

Proof. Since the order of G_1 is a power of p, we deduce from Theorem 1.1.17 that the R-groups G_1° and $G_1^{\text{ét}}$ respectively have order p^{h° and $p^{h^{\text{ét}}}$ for some integers h° and $h^{\text{ét}}$. Meanwhile, as Lemma 2.1.6 yields a natural isomorphism $G_{v+1}/G_v \cong G_1$ for each $v \geq 1$, we find $G_{v+1}^\circ/G_v^\circ \cong G_1^\circ$ and $G_{v+1}^{\text{ét}}/G_v^{\text{ét}} \cong G_1^{\text{ét}}$ by Proposition 1.4.14. A simple induction based on Theorem 1.1.17 shows that the R-groups G_v° and $G_v^{\text{ét}}$ respectively have order p^{vh° and $p^{vh^{\text{ét}}}$. In addition, Proposition 1.4.14 yields short exact sequences

$$\underline{0} \longrightarrow G_v^\circ \longrightarrow G_{v+1}^\circ \xrightarrow{[p^v]} G_{v+1}^\circ \qquad \text{and} \qquad \underline{0} \longrightarrow G_v^{\text{ét}} \longrightarrow G_{v+1}^{\text{ét}} \xrightarrow{[p^v]} G_{v+1}^{\text{ét}}.$$

Therefore $G^\circ = \varinjlim G_v^\circ$ and $G^{\text{ét}} = \varinjlim G_v^{\text{ét}}$ are p-divisible groups over R. Now the desired assertions are evident by Proposition 1.4.13 and Proposition 1.4.15. □

Remark. Proposition 2.1.11 implies an interesting fact that for a p-divisible group $G = \varinjlim G_v$ over a henselian local ring R each G_v being connected or étale is equivalent to G_1 being connected or étale.

Definition 2.1.12. Let $G = \varinjlim G_v$ be a p-divisible group over R.

(1) We say that G is *connected* if each G_v is connected.
(2) We say that G is *étale* if each G_v is étale.
(3) If R is a henselian local ring, we refer to the p-divisible groups G° and $G^{\text{ét}}$ in Proposition 2.1.11 respectively as the *connected part* and the *étale part* of G.

Example 2.1.13. Below are essential examples of étale or connected p-divisible groups.

(1) The constant p-divisible group $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is étale by Proposition 1.3.7.
(2) If R is a henselian local ring with residue field of characteristic p, the p-power roots of unity μ_{p^∞} is connected by Example 1.4.8.

Definition 2.1.14. Assume that $R = k$ is a field of characteristic p. Let $G = \varinjlim G_v$ be a p-divisible group over k and r be a positive integer.

(1) The p^r*-Frobenius twist* of G is $G^{(p^r)} := \varinjlim G_v^{(p^r)}$ with transition maps given by the p^r-Frobenius twists of the transition maps for G.
(2) We define the p^r*-Frobenius* of G to be $\varphi_G^{[r]} := (\varphi_{G_v}^{[r]})$ and the p^r*-Verschiebung* of G to be $\psi_G^{[r]} := (\psi_{G_v}^{[r]})$.
(3) For $r = 1$, we often refer to $\varphi_G := \varphi_G^{[1]}$ and $\psi_G := \psi_G^{[1]}$ respectively as the *Frobenius* and the *Verschiebung* of G.

PROPOSITION 2.1.15. Assume that $R = k$ is a field of characteristic p. Let G be a p-divisible group of height h over k and r be a positive integer.

(1) The ind-scheme $G^{(p^r)}$ is a p-divisible group of height h over k.
(2) The maps $\varphi_G^{[r]}$ and $\psi_G^{[r]}$ are homomorphisms of p-divisible groups.
(3) We have $\psi_G^{[r]} \circ \varphi_G^{[r]} = [p^r]_G$ and $\varphi_G^{[r]} \circ \psi_G^{[r]} = [p^r]_{G^{(p^r)}}$.

PROOF. The assertions are direct consequences of Proposition 1.5.8, Lemma 1.5.13, and Proposition 1.5.15. □

Remark. We can alternatively deduce the first statement from Lemma 2.1.5.

Definition 2.1.16. Assume that $R = k$ is a field. For a p-divisible group $G = \varinjlim G_v$ over k, we define its *Tate module* to be $T_p(G) := \varprojlim G_v(\overline{k})$ with transition maps induced by $[p]_G$.

Remark. We define $T_p(G)$ as an inverse limit of groups, while G is a direct limit of k-groups.

PROPOSITION 2.1.17. Assume that $R = k$ is a perfect field of characteristic not equal to p. There exists an equivalence of categories

$$\{\, p\text{-divisible groups over } k \,\} \xrightarrow{\sim} \{\, \text{finite free } \mathbb{Z}_p\text{-modules with a continuous } \Gamma_k\text{-action} \,\}$$

which sends each p-divisible group G over k to $T_p(G)$.

PROOF. Let $G = \varinjlim G_v$ be a p-divisible group over k. Lemma 2.1.6 implies that each $G_v(\overline{k})$ is a finite free module over $\mathbb{Z}/p^v\mathbb{Z}$. Moreover, each $G_v(\overline{k})$ naturally carries a continuous Γ_k-action. Hence $T_p(G) = \varprojlim G_v(\overline{k})$ is a finite free $\mathbb{Z}_p$-module with a continuous Γ_k-action.

Since all finite flat k-groups of p-power order are étale by Theorem 1.3.10, it is not hard to deduce from Proposition 1.3.4 that the functor is fully faithful. Moreover, given a finite free $\mathbb{Z}_p$-module M with a continuous Γ_k-action, Proposition 1.3.4 yields a finite étale k-group G_v with $G_v(\overline{k}) = M/(p^v)$ for each $v \geq 1$ and in turn provides a p-divisible group $G = \varinjlim G_v$ with $T_p(G) = M$. Therefore we deduce that the functor is an equivalence as desired. □

2.2. Serre-Tate equivalence for connected p-divisible groups

In this subsection, we introduce formal group laws and explore their relations to p-divisible groups. Throughout this subsection, we assume that R is a complete reduced noetherian local ring with residue field k of characteristic p and let $\mathscr{A}_d := R[[t_1, \cdots, t_d]]$ denote the ring of power series over R in d variables. We often write $\mathscr{A} := \mathscr{A}_d$ if the context clearly specifies d. We work with the canonical identifications $\mathscr{A}_d \widehat{\otimes}_R \mathscr{A}_d \cong R[[T, U]]$ and $\mathscr{A}_d \widehat{\otimes}_R \mathscr{A}_d \widehat{\otimes}_R \mathscr{A}_d \cong R[[T, U, V]]$, where we write $T := (t_1, \cdots, t_d)$, $U := (u_1, \cdots, u_d)$, and $V := (v_1, \cdots, v_d)$.

Lemma 2.2.1. An R-algebra homomorphism $f : R[[t_1, \cdots, t_n]] \to R[[u_1, \cdots, u_m]]$ is continuous if and only if each $f(t_i)$ lies in the ideal $\mathscr{I} := (u_1, \cdots, u_m)$.

Proof. The map f is continuous if and only if there exists an integer v with $f(t_i^v) \in \mathscr{I}$ for each $i = 1, \cdots, n$. Hence the assertion follows from our assumption that R is reduced. □

Definition 2.2.2. A *formal group law of dimension* d over R is a continuous R-algebra homomorphism $\mu : \mathscr{A}_d \to \mathscr{A}_d \widehat{\otimes}_R \mathscr{A}_d$ such that $\Phi(T, U) := (\mu(t_i))$ satisfies the following axioms:

(i) associativity axiom $\Phi(T, \Phi(U, V)) = \Phi(\Phi(T, U), V)$,
(ii) unit section axiom $\Phi(T, 0) = T = \Phi(0, T)$,
(iii) commutativity axiom $\Phi(T, U) = \Phi(U, T)$.

Example 2.2.3. We present two primary examples of one-dimensional formal group laws.

(1) The *additive formal group law* over R is the continuous R-algebra homomorphism $\mu_{\widehat{\mathbb{G}_a}} : R[[t]] \to R[[t, u]]$ with $\mu_{\widehat{\mathbb{G}_a}}(t) = t + u$.
(2) The *multiplicative formal group law* over R is the continuous R-algebra homomorphism $\mu_{\widehat{\mathbb{G}_m}} : R[[t]] \to R[[t, u]]$ with $\mu_{\widehat{\mathbb{G}_m}}(t) = t + u + tu = (1 + t)(1 + u) - 1$.

Lemma 2.2.4. Let $\mu : \mathscr{A} \to \mathscr{A} \widehat{\otimes}_R \mathscr{A}$ be a formal group law of dimension d over R represented by $\Phi(T, U) := (\mu(t_i))$. We have a d-tuple $\Xi(T) = (\Xi_i(T))$ of power series in d variables with

$$\Phi(T, \Xi(T)) = 0 = \Phi(\Xi(T), T).$$

Proof. By the commutativity axiom for μ, it suffices to construct a d-tuple $\Xi(T)$ with $\Phi(T, \Xi(T)) = 0$. Let us consider the ideal $\mathscr{I} := (t_1, \cdots, t_d)$ of $\mathscr{A}$. We have a natural identification $\mathscr{I} \widehat{\otimes} \mathscr{I} \cong (t_1, \cdots, t_d, u_1, \cdots, u_d)$. For each R-module M, we regard $M^{\times d}$ as the set of d-tuples whose entries all lie in M. We wish to present the desired d-tuple as a limit $\Xi(T) = \lim_{j \to \infty} P_j(T)$ where each $P_j(T)$ is a d-tuple of polynomials with

$$P_j(T) \in P_{j-1}(T) + (\mathscr{I}^j)^{\times d} \qquad \text{and} \qquad \Phi(P_j(T), T) \in (\mathscr{I}^{j+1})^{\times d}.$$

The unit section axiom for μ yields the relation

$$\Phi(T, U) \in T + U + ((\mathscr{I} \widehat{\otimes} \mathscr{I})^2)^{\times d}. \tag{2.2}$$

Let us set $P_1(T) := -T$ and inductively construct $P_j(T)$ for each $j > 1$. By the relation $\Phi(P_{j-1}(T), T) \in (\mathscr{I}^j)^{\times d}$, there exists a d-tuple $\Delta_j(T) \in (\mathscr{I}^j)^{\times d}$ with

$$\Delta_j(T) \in -\Phi(P_{j-1}(T), T) + (\mathscr{I}^{j+1})^{\times d}. \tag{2.3}$$

For $P_j(T) := P_{j-1}(T) + \Delta_j(T)$, we have $P_j(T) \in P_{j-1}(T) + (\mathscr{I}^j)^{\times d}$ and find

$$\Phi(P_j(T), T) = \Phi(P_{j-1}(T) + \Delta_j(T), T) \in \Phi(P_{j-1}(T), T) + \Delta_j(T) + (\mathscr{I}^{j+1})^{\times d} = (\mathscr{I}^{j+1})^{\times d}$$

by the relations (2.2) and (2.3). Therefore we obtain a desired d-tuple $\Xi(T)$. □

Remark. Lemma 2.2.4 shows that the inverse axiom is automatic for formal group laws.

LEMMA 2.2.5. Let $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ be a formal group law of dimension d over R.

(1) The formal group law μ yields commutative diagrams

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\ \mu\ } & \mathscr{A}\widehat{\otimes}_R\mathscr{A} \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu\widehat{\otimes}\mathrm{id}} \\ \mathscr{A}\widehat{\otimes}_R\mathscr{A} & \xrightarrow{\ \mathrm{id}\widehat{\otimes}\mu\ } & \mathscr{A}\widehat{\otimes}_R\mathscr{A}\widehat{\otimes}_R\mathscr{A} \end{array} \qquad \begin{array}{ccc} \mathscr{A}\widehat{\otimes}_R\mathscr{A} & \xrightarrow{\ x\widehat{\otimes}y\mapsto y\widehat{\otimes}x\ } & \mathscr{A}\widehat{\otimes}_R\mathscr{A} \\ {\scriptstyle\mu}\nwarrow & & \nearrow{\scriptstyle\mu} \\ & \mathscr{A} & \end{array}$$

(2) The R-algebra map $\epsilon : \mathscr{A} \to R$ with $\epsilon(t_i) = 0$ fits into commutative diagrams

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{\ \mathrm{id}\ } & \mathscr{A} & \xrightarrow{\ \sim\ } & \mathscr{A}\widehat{\otimes}_R R \\ & {\scriptstyle\mu}\searrow & & \nearrow{\scriptstyle\mathrm{id}\widehat{\otimes}\epsilon} & \\ & & \mathscr{A}\widehat{\otimes}_R\mathscr{A} & & \end{array} \qquad \begin{array}{ccccc} \mathscr{A} & \xrightarrow{\ \mathrm{id}\ } & \mathscr{A} & \xrightarrow{\ \sim\ } & R\widehat{\otimes}_R\mathscr{A} \\ & {\scriptstyle\mu}\searrow & & \nearrow{\scriptstyle\epsilon\widehat{\otimes}\mathrm{id}} & \\ & & \mathscr{A}\widehat{\otimes}_R\mathscr{A} & & \end{array}$$

(3) There exists an R-algebra map $\iota : \mathscr{A} \to \mathscr{A}$ that fits into a commutative diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\ \mu\ } & \mathscr{A}\widehat{\otimes}_R\mathscr{A} \\ {\scriptstyle\epsilon}\downarrow & & {\scriptstyle\mathrm{id}\widehat{\otimes}\iota}\downarrow\downarrow{\scriptstyle\iota\widehat{\otimes}\mathrm{id}} \\ R & \xrightarrow{\qquad} & \mathscr{A} \end{array}$$

PROOF. Statements (1) and (2) are evident by the axioms for μ. Statement (3) is a reformulation of Lemma 2.2.4. □

Remark. We can extend the notion of R-groups to define formal R-groups as group objects in the category of formal R-schemes. Lemma 2.2.5 shows that every formal group law μ of dimension d over R corresponds to a unique a formal R-group $\mathscr{G}_\mu = \mathrm{Spf}(\mathscr{A})$ with comultiplication μ, counit ϵ, and coinverse ι.

Definition 2.2.6. Let μ and ν be formal group laws over R.

(1) A *homomorphism* from μ and ν is a continuous R-algebra map $\theta : \mathscr{A}_{d'} \to \mathscr{A}_d$ with a commutative diagram

$$\begin{array}{ccc} \mathscr{A}_{d'} & \xrightarrow{\ \nu\ } & \mathscr{A}_{d'}\widehat{\otimes}_R\mathscr{A}_{d'} \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta\widehat{\otimes}\theta} \\ \mathscr{A}_d & \xrightarrow{\ \mu\ } & \mathscr{A}_d\widehat{\otimes}_R\mathscr{A}_d \end{array}$$

where d and d' respectively denotes the dimensions of μ and ν.

(2) A homomorphism $\theta : \mathscr{A}_{d'} \to \mathscr{A}_d$ from μ and ν is *finite flat* if $\mathscr{A}_d$ becomes a free module of finite rank over $\mathscr{A}_{d'}$ via θ.

Remark. The map θ goes from the power series ring for ν to the power series ring for μ so that it corresponds to a formal R-group homomorphism $\mathscr{G}_\mu \to \mathscr{G}_\nu$. If we consider the tuples $\Phi(T, U) := (\mu(t_i))$, $\Psi(T, U) := (\nu(t_j))$, and $\Xi(T) := (\theta(t_j))$, the commutative diagram for θ is equivalent to the identity $\Psi(\Xi(T), \Xi(U)) = \Xi(\Phi(T, T))$.

Example 2.2.7. Let μ be a formal group law of dimension d over R. For each integer $n \geq 1$, the *multiplication by* n on μ is the homomorphism $[n]_\mu : \mathscr{A} \to \mathscr{A}$ inductively defined by the relations $[1]_\mu := \mathrm{id}_{\mathscr{A}}$ and $[n]_\mu := ([n-1]_\mu\widehat{\otimes}\mathrm{id}) \circ \mu$.

Remark. The map $[n]_\mu$ induces the multiplication by n on the formal R-group $\mathscr{G}_\mu$.

Definition 2.2.8. Let $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ be a formal group law of dimension d over R.

(1) We refer to the ideal $\mathscr{I} := (t_1, \cdots, t_d)$ in $\mathscr{A}$ as the *augmentation ideal* of μ.
(2) We say that μ is *p-divisible* if the homomorphism $[p]_\mu : \mathscr{A} \to \mathscr{A}$ is finite flat.

Remark. The ideal $\mathscr{I}$ is the kernel of the counit $\epsilon : \mathscr{A} \to R$ for the formal R-group $\mathscr{G}_\mu$. Hence our definition here is comparable to the definition of augmentation ideal for affine R-groups.

Example 2.2.9. Let us consider the formal group laws from Example 2.2.3.

(1) The additive formal group law $\mu_{\widehat{\mathbb{G}_a}}$ is not p-divisible; indeed, $[p]_{\mu_{\widehat{\mathbb{G}_a}}}$ satisfies the identity $[p]_{\mu_{\widehat{\mathbb{G}_a}}}(t) = pt$ and thus is not finite flat for inducing a zero map on $\mathscr{A} \otimes_R k$.
(2) The multiplicative formal group law $\mu_{\widehat{\mathbb{G}_m}}$ is p-divisible; indeed, $[p]_{\mu_{\widehat{\mathbb{G}_m}}}$ satisfies the identity $[p]_{\mu_{\widehat{\mathbb{G}_m}}}(t) = (1+t)^p - 1$ and thus is finite flat.

Proposition 2.2.10. Let $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ be a p-divisible formal group law of dimension d over R with augmentation ideal $\mathscr{I}$. We write $A_v := \mathscr{A}/[p^v]_\mu(\mathscr{I})$ for each $v \geq 1$.

(1) Each $\mu[p^v] := \mathrm{Spec}\,(A_v)$ is naturally a connected finite flat R-group.
(2) The ind-scheme $\mu[p^\infty] := \varinjlim \mu[p^v]$ is a connected p-divisible group over R.

Proof. Let us take ϵ and ι as in Lemma 2.2.5. For each $v \geq 1$, we have

$$A_v = \mathscr{A}/[p^v]_\mu(\mathscr{I}) \cong \mathscr{A}/\mathscr{I} \otimes_{\mathscr{A},[p^v]_\mu} \mathscr{A} \cong R \otimes_{\mathscr{A},[p^v]_\mu} \mathscr{A}$$

and thus find that $\mu[p^v] = \mathrm{Spec}\,(A_v)$ is naturally an R-group with comultiplication $1\otimes\mu$, counit $1 \otimes \epsilon$, and coinverse $1 \otimes \iota$. If we take a basis of $\mathscr{A}$ over $[p]_\mu(\mathscr{A})$ given by $f_1, \cdots, f_r \in \mathscr{A}$, a simple induction yields a basis of $\mathscr{A}$ over $[p^v]_\mu(\mathscr{A})$ for each $v \geq 1$ given by elements of the form $[p^{v-1}]_\mu(f_{i_{v-1}}) \cdots [p]_\mu(f_{i_1})f_{i_0}$ with $(i_0, \cdots, i_{v-1}) \in (\mathbb{Z}/r\mathbb{Z})^v$ and consequently implies that $\mu[p^v]$ is finite flat of order r^v over R. Moreover, since R is a local ring, both $\mathscr{A}$ and $A_v = \mathscr{A}/[p^v]_\mu(\mathscr{I})$ are local rings as well. We deduce that $\mu[p^v]$ is connected and in turn establish statement (1).

Let us now consider statement (2). Lemma 1.4.3 and Proposition 1.5.20 together imply that $\mu[p]$ has order p^h for some integer h. Therefore our discussion in the previous paragraph shows that each $\mu[p^v]$ has order p^{vh}. Furthermore, the R-algebra homomorphism

$$A_v = \mathscr{A}/[p^v]_\mu(\mathscr{I}) \longrightarrow [p]_\mu(\mathscr{A})/[p^{v+1}]_\mu(\mathscr{I})$$

induced by $[p]_\mu$ is an isomorphism for being a surjective map between two free R-algebras of the same rank. Hence we obtain a surjective ring homomorphism

$$A_{v+1} = \mathscr{A}/[p^{v+1}]_\mu(\mathscr{I}) \twoheadrightarrow [p]_\mu(\mathscr{A})/[p^{v+1}]_\mu(\mathscr{I}) \simeq A_v,$$

which induces an embedding $i_v : \mu[p^v] \hookrightarrow \mu[p^{v+1}]$. Since it is evident by construction that i_v identifies $\mu[p^v]$ with the kernel of $[p^v]$ on $\mu[p^{v+1}]$, we conclude that $\mu[p^\infty] := \varinjlim \mu[p^v]$ is a connected p-divisible group of height h over R, thereby completing the proof. □

Remark. We can alternatively deduce statement (2) from statement (1) by the identifidcation $\mu[p^v] \cong \mathscr{G}_\mu[p^v]$ for each $v \geq 1$.

Definition 2.2.11. Given a p-divisible formal group law μ over R, we define its *associated connected p-divisible group* over R to be $\mu[p^\infty]$ as constructed in Proposition 2.2.10.

Example 2.2.12. The multiplicative formal group law $\mu_{\widehat{\mathbb{G}_m}}$ is p-divisible as explained in Example 2.2.9. For each $v \geq 1$, we have $[p^v]_{\mu_{\widehat{\mathbb{G}_m}}}(t) = (1+t)^{p^v} - 1$ and thus find $\mu_{\widehat{\mathbb{G}_m}}[p^v] \cong \mu_{p^v}$ by Example 1.1.8. Hence we obtain a natural identification $\mu_{\widehat{\mathbb{G}_m}}[p^\infty] \cong \mu_{p^\infty}$.

Our main objective for this subsection is to prove a theorem of Serre and Tate that the association described in Proposition 2.2.10 defines an equivalence between the category of p-divisible formal group laws and the category of connected p-divisible groups.

LEMMA 2.2.13. Let $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ be a formal group law of dimension d over R with augmentation ideal $\mathscr{I}$. For each integer $n \geq 1$, we have

$$[n]_\mu(t_i) \in nt_i + \mathscr{I}^2.$$

PROOF. Let us take d-tuples $\Phi(T, U) := (\mu(t_i))$ and $\Xi_n(T) := ([n]_\mu(t_i))$ for each $n \geq 1$. Given an R-module M, we regard $M^{\times d}$ as the set of d-tuples whose entries all lie in M. Under the natural identification $\mathscr{I}\widehat{\otimes}\mathscr{I} \cong (t_1, \cdots, t_d, u_1, \cdots, u_d)$, we find

$$\Phi(T, U) \in T + U + ((\mathscr{I}\widehat{\otimes}\mathscr{I})^2)^{\times d}.$$

by the unit section axiom for μ. Hence the identity $[n]_\mu = ([n-1]_\mu\widehat{\otimes}\mathrm{id}) \circ \mu$ yield the relation

$$\Xi_n(T) = \Phi(\Xi_{n-1}(T), T) \in \Xi_{n-1}(T) + T + (\mathscr{I}^2)^{\times d}.$$

Since we have $\Xi_1(T) = T$ by definition, we proceed by induction to find $\Xi_n(T) \in nT + (\mathscr{I}^2)^{\times d}$ for each $n \geq 1$, thereby completing the proof. □

Remark. The proof of Theorem 1.3.10 yields an analogous relation for finite flat R-groups.

LEMMA 2.2.14. Given a p-divisible formal group law $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ of dimension d over R with augmentation ideal $\mathscr{I}$, there exists a natural homeomorphic R-algebra isomorphism

$$\mathscr{A} \cong \varprojlim A_v$$

where we write $A_v := \mathscr{A}/[p^v]_\mu(\mathscr{I})$ for each $v \geq 1$.

PROOF. Since R is a local ring, $\mathscr{A}$ and A_v are also local rings for each $v \geq 1$. Moreover, each A_v is a free R-algebra of finite rank as noted in Proposition 2.2.10. Let us write $\mathfrak{m}$ for the maximal ideal of R and $\mathfrak{M} := \mathfrak{m}\mathscr{A} + \mathscr{I}$ for the maximal ideal of $\mathscr{A}$. We have $[p]_\mu(\mathscr{I}) \subseteq p\mathscr{I} + \mathscr{I}^2 \subseteq \mathfrak{M}\mathscr{I}$ by Lemma 2.2.13 and thus find $[p^v]_\mu(\mathscr{I}) \subseteq \mathfrak{M}^v\mathscr{I}$ for each $v \geq 1$. Hence for each $i,\ v \geq 1$ we have $[p^v]_\mu(\mathscr{I}) + \mathfrak{m}^i\mathscr{A} \subseteq \mathfrak{M}^w$ for some $w \geq 1$. Meanwhile, for each $i,\ v \geq 1$ we find $\mathfrak{M}^{w'} \subseteq [p^v]_\mu(\mathscr{I}) + \mathfrak{m}^i\mathscr{A}$ for some $w' \geq 1$ as $\mathscr{A}/([p^v]_\mu(\mathscr{I}) + \mathfrak{m}^i\mathscr{A}) = A_v/\mathfrak{m}^iA_v$ is local artinian. Now we obtain a homeomorphic R-algebra isomorphism

$$\mathscr{A} \cong \varprojlim_w \mathscr{A}/\mathfrak{M}^w \cong \varprojlim_{i,v} \mathscr{A}/([p^v]_\mu(\mathscr{I}) + \mathfrak{m}^i\mathscr{A}) \cong \varprojlim_{v,i} A_v/\mathfrak{m}^iA_v \cong \varprojlim_v A_v$$

where the last identification comes from an observation that each A_v is $\mathfrak{m}$-adically complete by a general fact stated in the Stacks project [**Sta**, Tag 031B]. □

LEMMA 2.2.15. Given p-divisible formal group laws μ and ν over R, there exists a natural identification

$$\mathrm{Hom}(\mu, \nu) \cong \mathrm{Hom}(\mu[p^\infty], \nu[p^\infty]).$$

PROOF. Let us write d and d' respectively for the dimensions of μ and ν. In addition, we set $A_v := \mathscr{A}_d/[p^v]_\mu(\mathscr{I}_\mu)$ and $B_v := \mathscr{A}_{d'}/[p^v]_\nu(\mathscr{I}_\nu)$ for each $v \geq 1$, where $\mathscr{I}_\mu$ and $\mathscr{I}_\nu$ respectively denote the augmentation ideals of μ and ν. Proposition 2.2.10 shows that $\mu[p^v] := \mathrm{Spec}\,(A_v)$ and $\nu[p^v] := \mathrm{Spec}\,(B_v)$ are connected finite flat R-groups. Since we have $\mathscr{A}_d \cong \varprojlim A_v$ and $\mathscr{A}_{d'} \cong \varprojlim B_v$ by Lemma 2.2.14, we obtain a natural identification

$$\mathrm{Hom}(\mu, \nu) \cong \varprojlim \mathrm{Hom}_{\nu_v,\mu_v}(B_v, A_v) \cong \varinjlim \mathrm{Hom}_{R\text{-grp}}(\mu[p^v], \nu[p^v]) = \mathrm{Hom}(\mu[p^\infty], \nu[p^\infty])$$

where $\mathrm{Hom}_{\nu_v,\mu_v}(B_v, A_v)$ denotes the set of R-algebra maps $B_v \to A_v$ compatible with the comultiplications μ_v on $\mu[p^v]$ and ν_v on $\nu[p^v]$. □

Proposition 2.2.16. Let $G = \varinjlim G_v$ be a connected p-divisible group over R.

(1) There exists a homeomorphic k-algebra isomorphism
$$\varprojlim (A_v \otimes_R k) \simeq k[[t_1, \cdots, t_d]] \quad \text{for some } d \geq 0$$
where A_v denotes the affine ring of G_v.

(2) The special fiber $\overline{G} := G \times_R k$ is a p-divisible group over k such that $\ker(\varphi_{\overline{G}})$ is a finite flat k-group of order p^d.

Proof. It is evident by Lemma 2.1.5 that $\overline{G}$ is a p-divisible group over k. Let us write $\overline{G}_v := G_v \times_R k$ and $H_v := \ker(\varphi_{\overline{G}}^{[v]})$ for each $v \geq 1$. Proposition 2.1.7 and Proposition 2.1.15 together imply that each H_v is a closed k-subgroup of $\overline{G}[p^v] \cong \overline{G}_v$. Moreover, each $\overline{G}_v$ is connected by Lemma 1.4.3 and thus is a k-subgroup of $\ker(\varphi_{\overline{G}}^{[w]}) = H_w$ for some $w \geq 1$ by Proposition 1.5.17. Therefore we write $H_v = \operatorname{Spec}(B_v)$ for each $v \geq 1$ and obtain a homeomorphic k-algebra isomorphism
$$\varprojlim A_v \otimes_R k \simeq \varprojlim B_v. \tag{2.4}$$

We denote the augmentation ideal of H_v by J_v and set $J := \varprojlim J_v$. Since each H_v is connected, as easily seen by Proposition 1.4.14 or Proposition 1.5.17, its affine ring B_v is a local k-algebra with maximal ideal J_v by Lemma 1.5.16. In addition, we have $H_1 \cong \ker(\varphi_{H_v})$ by Lemma 1.5.7 and thus apply Proposition 1.5.19 to obtain an isomorphism $B_1 \cong B_v/J_v^{(p)}$ where $J_v^{(p)}$ denotes the ideal generated by the p-th powers of elements in J_v. We find $J_1 \cong J_v/J_v^{(p)}$ and in turn get an identification $J_1/J_1^2 \cong J_v/J_v^2$. Let us take $b_1, \cdots, b_d \in J$ whose images in J_1/J_1^2 form a basis over k. Nakayama's lemma implies that J_v admits generators given by the images of $b_1, \cdots, b_d$ and in turn yields a surjective k-algebra homomorphism $k[t_1, \cdots, t_d] \twoheadrightarrow B_v$ which sends each t_i to the image of b_i in B_v. Furthermore, as $\varphi_{H_v}^{[v]}$ vanishes by Lemma 1.5.7, this map induces a surjective k-algebra homomorphism
$$\lambda_v : k[t_1, \cdots, t_d]/(t_1^{p^v}, \cdots, t_d^{p^v}) \twoheadrightarrow B_v$$
by Proposition 1.5.19. Therefore we obtain a continuous k-algebra homomorphism
$$\lambda : k[[t_1, \cdots, t_d]] \twoheadrightarrow \varprojlim B_v$$
via the identification $k[[t_1, \cdots, t_d]] \cong \varprojlim k[t_1, \cdots, t_d]/(t_1^{p^v}, \cdots, t_d^{p^v})$.

In light of the isomorphism (2.4), we wish to show that λ is a homeomorphism. It suffices to prove that each λ_v is an isomorphism. Since each λ_v is surjective by construction, we only need to verify that its source and target have equal dimensions over k; in other words, it is enough to show that B_v has dimension p^{dv} over k, or equivalently that H_v has order p^{dv}.

For $v = 1$, the assertion follows from Proposition 1.5.19. Let us henceforth assume $v > 1$ and proceed by induction. Proposition 2.1.15 shows that $\overline{G}^{(p)}$ is a p-divisible group over k with $\varphi_{\overline{G}} \circ \psi_{\overline{G}} = [p]_{\overline{G}^{(p)}}$. Since $[p]_{\overline{G}^{(p)}}$ is surjective as noted in Proposition 2.1.7, the map $\varphi_{\overline{G}}$ is also surjective and thus maps $H_v = \ker(\varphi_{\overline{G}}^{[v]})$ surjectively onto $\ker(\varphi_{\overline{G}^{(p)}}^{[v-1]}) \cong H_{v-1}^{(p)}$. We deduce that there exists a short exact sequence
$$\underline{0} \longrightarrow H_1 \longrightarrow H_v \longrightarrow H_{v-1}^{(p)} \longrightarrow \underline{0}.$$
Now the desired assertion follows from Theorem 1.1.17 and the fact that the order of $H_{v-1}^{(p)}$ is the same as the order of H_{v-1}. □

LEMMA 2.2.17. Given an R-algebra B, its ideal J with $J \otimes_R k = 0$ is trivial if for each maximal ideal $\mathfrak{n}$ of B the $B_\mathfrak{n}$-module $J_\mathfrak{n}$ admits a finite set of generators.

PROOF. Let us write $\mathfrak{m}$ for the maximal ideal of R. For each maximal ideal $\mathfrak{n}$ of B, we have $J_\mathfrak{n} = \mathfrak{m}J_\mathfrak{n} \subseteq \mathfrak{n}J_\mathfrak{n}$ and thus deduce from Nakayama's lemma that $J_\mathfrak{n}$ is trivial. □

LEMMA 2.2.18. Let $G = \varinjlim G_v$ be a p-divisible group over R with $G_v = \mathrm{Spec}\,(A_v)$.

(1) G gives rise to a flat R-algebra $\varprojlim A_v$.

(2) If an R-algebra B admits a k-algebra isomorphism
$$\overline{\theta} : (B \otimes_R k)[[t_1, \cdots, t_d]] \xrightarrow{\sim} \varprojlim (A_v \otimes_R k) \quad \text{for some } d \geq 0,$$
there exists an R-algebra surjection $\theta : B[[t_1, \cdots, t_d]] \longrightarrow \varprojlim A_v$ which lifts $\overline{\theta}$.

PROOF. Since each $i_v : G_v \to G_{v+1}$ is a closed embedding by Proposition 1.1.10, the induced map $\pi_v : A_{v+1} \to A_v$ is surjective. Hence statement (1) follows from a general fact stated in the Stacks project [**Sta**, Tag 0912]. It remains to establish statement (2).

We assert that each $\overline{\theta}_v : (B \otimes_R k)[[t_1, \cdots, t_d]] \twoheadrightarrow A_v \otimes_R k$ lifts to an R-algebra homomorphism $\theta_v : B[[t_1, \cdots, t_d]] \to A_v$ with a commutative diagram

$$\begin{array}{ccccc} B[[t_1, \cdots, t_d]] & \xrightarrow{\theta_{v+1}} & A_{v+1} & \longrightarrow & A_{v+1} \otimes_R k \\ & \searrow^{\theta_v} & \downarrow\!\!\downarrow \pi_v & & \downarrow\!\!\downarrow \pi_v \otimes \mathrm{id} \\ & & A_v & \longrightarrow & A_v \otimes_R k \end{array}$$

We take θ_1 to be an arbitrary lift of $\overline{\theta}_1$ and proceed by induction on v. Let us write $\mathfrak{m}$ for the maximal ideal of R and choose $a_1, \cdots, a_d \in A_{v+1}$ which lift $\overline{\theta}_{v+1}(t_1), \cdots, \overline{\theta}_{v+1}(t_d)$. We observe that $\pi_v(a_1), \cdots, \pi_v(a_d)$ lift $\overline{\theta}_v(t_1), \cdots, \overline{\theta}_v(t_d)$ and in turn find $\theta_v(t_i) - \pi_v(a_i) \in \mathfrak{m}A_v$. Since π_v is surjective, we may choose $b_1, \cdots, b_d \in \mathfrak{m}A_{v+1}$ with $\pi_v(b_i) = \theta_v(t_i) - \pi_v(a_i)$ and deduce that $\overline{\theta}_{v+1}$ lifts to a map $\theta_{v+1} : B[[t_1, \cdots, t_d]] \to A_{v+1}$ with $\theta_{v+1}(t_i) = a_i + b_i$ as desired.

Now we have an R-algebra homomorphism $\theta : B[[t_1, \cdots, t_d]] \longrightarrow \varprojlim A_v$ which lifts $\overline{\theta}$. We find $\mathrm{coker}(\theta) \otimes_R k = \mathrm{coker}(\overline{\theta}) = 0$ and also observe that $\mathrm{coker}(\theta)$ admits a generator over $\varprojlim A_v$ given by the image of 1. Therefore Lemma 2.2.17 implies that θ is surjective. □

LEMMA 2.2.19. Every connected p-divisible group $G = \varinjlim G_v$ over R with $G_v = \mathrm{Spec}\,(A_v)$ yields a formal group law $\mu : \mathscr{A} \to \mathscr{A} \widehat{\otimes}_R \mathscr{A}$ via a homeomorphic R-algebra isomorphism
$$\mathscr{A} = R[[t_1, \cdots, t_d]] \simeq \varprojlim A_v \quad \text{for some } d \geq 0.$$

PROOF. Proposition 2.2.16 and Lemma 2.2.18 yield a surjective R-algebra homomorphism $\theta : \mathscr{A} \twoheadrightarrow \varprojlim A_v$ which lifts a homeomorphic isomorphism $\overline{\theta} : k[[t_1, \cdots, t_d]] \xrightarrow{\sim} \varprojlim (A_v \otimes_R k)$. In addtion, we have $\ker(\theta) \otimes_R k = \ker(\overline{\theta}) = 0$ by Lemma 2.2.18 and a general fact stated in the Stacks project [**Sta**, Tag 00HL]. Since $\mathscr{A}$ is a notherian local ring, we find $\ker(\theta) = 0$ by Lemma 2.2.17 and in turn deduce that θ is an isomorphism.

The map θ is continuous as the kernel of each $\theta_v : \mathscr{A} \to A_v$ is open by the fact that the R-algebra A_v is of finite length. Moreover, with $\overline{\theta}$ being a homeomorphism we observe that every power of the ideal $\mathscr{I} := (t_1, \cdots, t_d)$ contains an open set in its image under θ and consequently find that θ is open. Therefore θ is a homeomorphic R-algebra isomorphism.

Let us denote the comultiplication of each G_v by μ_v. Via the isomorphism θ we may identify $\varprojlim \mu_v$ with a continuous R-algebra homomorphism $\mu : \mathscr{A} \to \mathscr{A} \widehat{\otimes}_R \mathscr{A}$. It is evident by the axioms for each comultiplication μ_v that μ is indeed a formal group law over R. □

THEOREM 2.2.20 (Serre-Tate). There exists an equivalence of categories

$$\{ \text{p-divisible formal group laws over } R \} \xrightarrow{\sim} \{ \text{connected p-divisible groups over } R \}$$

which maps each p-divisible formal group law μ over R to $\mu[p^\infty]$.

PROOF. Since Lemma 2.2.15 shows that the functor is fully faithful, we only need to prove that the functor is essentially surjective. Let $G = \varinjlim G_v$ be an arbitrary connected p-divisible group of height h over R with $G_v = \operatorname{Spec}(A_v)$. Lemma 2.2.19 yields a formal group law $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ induced by G via a homeomorphic R-algebra isomorphism

$$\mathscr{A} = R[[t_1, \cdots, t_d]] \simeq \varprojlim A_v \quad \text{ for some } d \geq 0.$$

We wish to show that μ is p-divisible with $\mu[p^\infty] \simeq G$.

We denote the agumentation ideal of $\mathscr{A}$ by $\mathscr{I}$. For each $v \geq 1$, we have $G_v \cong \ker([p^v]_G)$ by Proposition 2.1.7 and thus find $A_v \simeq \mathscr{A}/[p^v]_\mu(\mathscr{I})$. Let us write $r := p^h$ and choose $f_1, \cdots, f_r \in \mathscr{A}$ whose images in $A_1 \simeq \mathscr{A}/[p]_\mu(\mathscr{I})$ form a basis over R.

For every $g \in \mathscr{A}$, a simple induction yields a sequence $(g_{i,j})$ for each $i = 1, \cdots, r$ with

$$g_{i,j} \in g_{i,j-1} + \mathscr{I}^{j-1} \qquad \text{and} \qquad g \in \sum_{i=1}^{r} [p]_\mu(g_{i,j}) f_i + [p]_\mu(\mathscr{I})^j.$$

Since we have $[p]_\mu(\mathscr{I}) \subseteq \mathscr{I}$ by Lemma 2.2.1, we set $g_i := \lim\limits_{j\to\infty} g_{i,j}$ and find $g = \sum\limits_{i=1}^{r} [p]_\mu(g_i) f_i$. Hence we deduce that $f_1, \cdots, f_r$ generate $\mathscr{A}$ over $[p]_\mu(\mathscr{A})$.

As noted in Lemma 2.1.6, each $[p]_{G_v}$ factors through a surjective R-group homomorphism $j_v : G_{v+1} \twoheadrightarrow G_v$, which in turn induces a faithfully flat R-algebra homomorphism

$$\eta_v : A_v \simeq \mathscr{A}/[p^v]_\mu(\mathscr{I}) \longrightarrow \mathscr{A}/[p^{v+1}]_\mu(\mathscr{I}) \simeq A_{v+1}$$

by a standard fact stated in the Stacks project [**Sta**, Tag 00HQ]. Since each A_v is a free local R-algebra of rank p^{vh}, we see that A_{v+1} is free over A_v of rank $r = p^h$ and in turn deduce that the images of $f_1, \cdots, f_r$ in $A_{v+1} \simeq \mathscr{A}/[p^{v+1}]_\mu(\mathscr{I})$ form a basis over $A_v \simeq \mathscr{A}/[p^v]_\mu(\mathscr{I})$.

Let us now consider a relation $\sum\limits_{i=1}^{r} [p]_\mu(h_i) f_i = 0$ with $h_1, \cdots, h_r \in \mathscr{A}$. For each $v \geq 1$, we consider this relation in $A_{v+1} \simeq \mathscr{A}/[p^{v+1}]_\mu(\mathscr{I})$ and find $[p]_\mu(h_1), \cdots, [p]_\mu(h_r) \in [p^v]_\mu(\mathscr{I})$. Since we have $[p^v]_\mu(\mathscr{I}) \subseteq \mathscr{I}^v$ for each $v \geq 1$ as easily seen by Lemma 2.2.1, we deduce that $[p]_\mu(h_1), \cdots, [p]_\mu(h_r)$ must all be zero. Therefore we find that $f_1, \cdots, f_r$ form a basis of $\mathscr{A}$ over $[p]_\mu(\mathscr{A})$, which in particular implies that μ is p-divisible. As we evidently have $\mu[p^\infty] \simeq G$ by construction, we deduce the desired assertion and complete the proof. □

Remark. Our proof yields a formal R-group isomorphism $\mathscr{G}_\mu \simeq \varinjlim G_v$ with $\mathscr{G}_\mu[p^v] \simeq G_v$.

Definition 2.2.21. Let G be a p-divisible group over R.

(1) We define its *associated formal group law* to be the p-divisible formal group law μ_G over R corresponding to G° under the equivalence in Theorem 2.2.20.
(2) We define its *dimension* to be the dimension of μ_G.

PROPOSITION 2.2.22. Given a p-divisible group G over R of dimension d, its special fiber $\overline{G} := G \times_R k$ is a p-divisible group over k such that $\ker(\varphi_{\overline{G}})$ is finite flat of order p^d.

PROOF. Proposition 1.5.17 implies that $\ker(\varphi_{\overline{G}})$ lies in $\overline{G}^\circ := G^\circ \times_R k$. Hence the assertion follows from Proposition 2.2.16, Lemma 2.2.19, and Theorem 2.2.20. □

THEOREM 2.2.23. Let G be a p-divisible group of height h over R. If write d and $d^\vee$ respectively for the dimensions of G and $G^\vee$, we have $h = d + d^\vee$.

PROOF. Lemma 2.1.5 shows that $\overline{G} := G \times_R k$ is a p-divisible group of order h over k. Let us write $\overline{G} = \varinjlim \overline{G}_v$ where each $\overline{G}_v$ is a finite flat k-group scheme. We have $\psi_{\overline{G}} \circ \varphi_{\overline{G}} = [p]_{\overline{G}}$ as noted in Proposition 2.1.15 and thus find $\ker(\varphi_{\overline{G}}) \subseteq \overline{G}[p]$. In addition, we deduce from Proposition 2.1.7 that $\varphi_{\overline{G}}$ is surjective. Therefore we obtain a commutative diagram

$$\begin{array}{ccccccccc} \underline{0} & \longrightarrow & \ker(\varphi_{\overline{G}}) & \longrightarrow & \overline{G} & \xrightarrow{\varphi_{\overline{G}}} & \overline{G}^{(p)} & \longrightarrow & \underline{0} \\ & & \downarrow & & \downarrow{\scriptstyle [p]_{\overline{G}}} & & \downarrow{\scriptstyle \psi_{\overline{G}}} & & \\ \underline{0} & \longrightarrow & \underline{0} & \longrightarrow & \overline{G} & \xrightarrow{\mathrm{id}} & \overline{G} & \longrightarrow & \underline{0} \end{array}$$

where the rows are evidently exact. By the snake lemma, the diagram yields an exact sequence

$$\underline{0} \longrightarrow \ker(\varphi_{\overline{G}}) \longrightarrow \overline{G}[p] \longrightarrow \ker(\psi_{\overline{G}}) \longrightarrow \underline{0}.$$

Proposition 2.2.22 shows that $\ker(\varphi_{\overline{G}})$ has order p^d, while Proposition 2.1.7 implies that $\overline{G}[p] \cong G_1$ has order p^h. Hence we deduce from Theorem 1.1.17 that $\ker(\psi_{\overline{G}})$ has order p^{h-d}.

For the desired assertion, it suffices to show that $\ker(\psi_{\overline{G}})$ has order $p^{d^\vee}$. We have

$$\ker(\psi_{\overline{G}}) \cong \ker(\psi_{\overline{G}_1}) \qquad \text{and} \qquad \ker(\varphi_{\overline{G}^\vee}) \cong \ker(\varphi_{\overline{G}_1^\vee})$$

as easily seen by Proposition 2.1.7 and Proposition 2.1.15. Since the k-groups $\overline{G}_1$ and $\overline{G}_1^{(p)}$ are of the same order by construction, we apply Theorem 1.1.17 with the identifications

$$\psi_{\overline{G}_1}(\overline{G}_1^{(p)}) \cong \overline{G}_1^{(p)} / \ker(\psi_{\overline{G}_1}) \qquad \text{and} \qquad \operatorname{coker}(\psi_{\overline{G}_1}) \cong \overline{G}_1 / \psi_{\overline{G}_1}(\overline{G}_1^{(p)})$$

to find that $\ker(\psi_{\overline{G}_1})$ and $\operatorname{coker}(\psi_{\overline{G}_1})$ are of the same order. Moreover, Proposition 1.2.13 yields a natural isomorphism $\operatorname{coker}(\psi_{\overline{G}_1}) \cong \ker(\varphi_{\overline{G}_1^\vee})$ as we have $\psi_{\overline{G}_1} = \varphi_{\overline{G}_1^\vee}^\vee$ by definition. Therefore $\ker(\psi_{\overline{G}})$ and $\ker(\varphi_{\overline{G}^\vee})$ have the same order. Since we have $\overline{G}^\vee \cong G^\vee \times_R k$ by Proposition 1.2.5, we deduce from Proposition 2.2.22 that $\ker(\psi_{\overline{G}^\vee})$ has order $p^{d^\vee}$, thereby establishing the desired assertion. □

PROPOSITION 2.2.24. Assume that $R = k$ is an algebraically closed field of characteristic p. Every p-divisible group $G = \varprojlim G_v$ of height 1 over k is isomorphic to either $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ or μ_{p^∞}.

PROOF. Let us first consider the case where G is étale. Each G_v is a finite étale k-group of order p^v with $G_v = G_{v+1}[p^v]$. Since every finite étale k-group is a constant group scheme as noted in Proposition 1.3.8, we find $G_v \simeq \underline{\mathbb{Z}/p^v\mathbb{Z}}$ for each $v \geq 1$ by a simple induction and in turn obtain an isomorphism $G \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p}$.

We now turn to the case where G is not étale. A p-divisible group over R is étale if and only if it has dimension 0, as easily seen by Proposition 2.1.11. Since G has height 1, we deduce from Theorem 2.2.23 that $G^\vee$ is étale and thus find $G^\vee \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p}$. Hence we obtain an isomorphism $G \simeq (\underline{\mathbb{Q}_p/\mathbb{Z}_p})^\vee \simeq \mu_{p^\infty}$ by Proposition 2.1.8 and Example 2.1.10, thereby completing the proof. □

Example 2.2.25. Let E be an ordinary elliptic curve over $\overline{\mathbb{F}}_p$. Example 1.4.16 shows that both $E[p^\infty]^\circ$ and $[p^\infty]^{\text{ét}}$ are of height 1 with $E[p]^\circ \simeq \mu_p$ and $E[p]^{\text{ét}} \simeq \underline{\mathbb{Z}/p\mathbb{Z}}$. Therefore Proposition 2.1.11 and Proposition 2.2.24 together yield an isomorphism

$$E[p^\infty] \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p} \times \mu_{p^\infty}.$$

2.3. Dieudonné-Manin classification

Throughout this subsection, we assume that $R = k$ is a perfect field of characteristic p. We introduce several algebraic objects and discuss their relation to p-divisible groups over k. We begin with a brief overview on Witt vectors where we omit some technical details.

THEOREM 2.3.1. *Let A be a perfect $\mathbb{F}_p$-algebra.*

(1) There exists a unique (up to isomorphism) ring $W(A)$ which is p-adically complete with $W(A)/pW(A) \cong A$.

(2) Given a p-adically complete ring B, every homomorphism $\overline{f} : A \to B/pB$ uniquely lifts to a multiplicative map $\hat{f} : A \to B$ and a homomorphism $f : W(A) \to B$.

Remark. For a proof, we refer readers to the book of Serre [**Ser79**, §II.5].

Definition 2.3.2. Let A be a perfect $\mathbb{F}_p$-algebra.

(1) We refer to the ring $W(A)$ in Theorem 2.3.1 as the *ring of Witt vectors over A*.

(2) For each $a \in A$, we define its *Teichmüller lift* $[a] \in W(A)$ to be its image under the unique multiplicative map $A \to W(A)$ which lifts the identity map on A.

Example 2.3.3. We present two important examples which frequently arise in practice.

(1) For $q = p^r$ with an integer $r \geq 1$, the ring $W(\mathbb{F}_q)$ is isomorphic to the valuation ring of the unramified extension of degree r over $\mathbb{Q}_p$, as easily seen by Theorem 2.3.1.

(2) The ring $W(\overline{\mathbb{F}}_p)$ is the valuation ring of $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$, where $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ denotes the completion of the maximal unramified extension of $\mathbb{Q}_p$.

PROPOSITION 2.3.4. Let A be a perfect $\mathbb{F}_p$-algebra.

(1) For every $\alpha \in W(A)$, there exists a unique element $a_0 \in A$ with $\alpha - [a_0] \in pW(A)$.

(2) Every $\alpha \in W(A)$ admits a unique expression $\alpha = \displaystyle\sum_{n=0}^{\infty} [a_n]p^n$ with $a_n \in A$.

(3) The p-th power map on A uniquely lifts to an automorphism $\varphi_{W(A)}$ on $W(A)$ with

$$\varphi_{W(A)}\left(\sum_{n=0}^{\infty}[a_n]p^n\right) = \sum_{n=0}^{\infty}[a_n^p]p^n.$$

PROOF. Statement (1) is evident with a_0 given by the image of α under the natural map $W(A) \twoheadrightarrow W(A)/pW(A) \cong A$. Statement (2) follows from statement (1) by inductively constructing a unique sequence (a_n) in A with

$$\alpha - \sum_{n=0}^{m}[a_n]p^n \in p^m W(A) \quad \text{for each } m \geq 0.$$

Statement (3) is straightforward to verify by Theorem 2.3.1 and the perfectness of A. □

Definition 2.3.5. Let A be a perfect $\mathbb{F}_p$-algebra.

(1) For every $\alpha \in W(A)$, we define its *Teichmüller expansion* to be the unique expression $\alpha = \displaystyle\sum_{n=0}^{\infty}[a_n]p^n$ with $a_n \in A$ given by Proposition 2.3.4.

(2) We call the map $\varphi_{W(A)}$ in Proposition 2.3.4 the *Frobenius automorphism* of $W(A)$.

Remark. Teichmüller expansions for $\mathbb{Z}_p = W(\mathbb{F}_p)$ are not the same as p-adic expansions.

PROPOSITION 2.3.6. Let A be a perfect $\mathbb{F}_p$-algebra. Take two arbitrary elements $\alpha,\ \beta \in W(A)$ with Teichmüller expansions $\alpha = \sum_{n=0}^{\infty}[a_n]p^n$ and $\beta = \sum_{n=0}^{\infty}[b_n]p^n \in W(A)$.

(1) The Teichmüller expansion of $\alpha+\beta$ has the first two coefficients given by

$$c_0 = a_0 + b_0 \quad \text{and} \quad c_1 = a_1 + b_1 - W_1\left(a_0^{1/p}, b_0^{1/p}\right),$$

where we write $W_1(t,u) := \dfrac{(t+u)^p - t^p - u^p}{p} \in \mathbb{Z}[t,u]$.

(2) The Teichmüller expansion of $\alpha\beta$ has the first two coefficients given by

$$d_0 = a_0 b_0 \quad \text{and} \quad d_1 = a_0 b_1 + a_1 b_0.$$

PROOF. The addition under the natural surjection $W(A) \twoheadrightarrow W(A)/pW(A) \cong A$ yields the identity $c_0 = a_0 + b_0$. Since every element of A admits a unique p-th root, we have $c_0^{1/p} = a_0^{1/p} + b_0^{1/p}$. Hence we find $[c_0^{1/p}] \in [a_0^{1/p}] + [b_0^{1/p}] + pW(A)$ and in turn get the relation

$$[c_0] = [c_0^{1/p}]^p \in \left([a_0^{1/p}] + [b_0^{1/p}]\right)^p + p^2 W(A).$$

Meanwhile, the addition under the natural map $W(A) \twoheadrightarrow W(A)/p^2W(A)$ yields the relation

$$[c_0] + p[c_1] = [a_0] + [b_0] + p([a_1] + [b_1]) + p^2 W(A).$$

Now we have

$$p[c_1] \in p([a_1] + [b_1]) + [a_0] + [b_0] - \left([a_0^{1/p}] + [b_0^{1/p}]\right)^p + p^2 W(A)$$

and consequently find

$$[c_1] \in [a_1] + [b_1] - W_1\left([a_0^{1/p}], [b_0^{1/p}]\right) + pW(A).$$

We consider the images under the natural surjection $W(A) \twoheadrightarrow W(A)/pW(A) \cong A$ and obtain the identity $c_1 = a_1 + b_1 - W_1\left(a_0^{1/p}, b_0^{1/p}\right)$. Therefore we establish statement (1).

Let us now consider statement (2). The multiplication under the natural surjection $W(A) \twoheadrightarrow W(A)/pW(A) \cong A$ yields the identity $d_0 = a_0 b_0$. Moreover, the multiplication under the natural map $W(A) \twoheadrightarrow W(A)/p^2W(A)$ yields the relation

$$[d_0] + p[d_1] \in [a_0 b_0] + p([a_0 b_1] + [a_1 b_0]) + p^2 W(A).$$

Hence we have

$$p[d_1] \in p([a_0 b_1] + [a_1 b_0]) + p^2 W(A)$$

and consequently find

$$[d_1] \in [a_0 b_1] + [a_1 b_0] + pW(A).$$

We consider the images under the natural surjection $W(A) \twoheadrightarrow W(A)/pW(A) \cong A$ and deduce the identity $d_1 = a_0 b_1 + a_1 b_0$, thereby completing the proof. □

Remark. We can inductively proceed to express the n-th coefficients in the Teichmüller expansion of $\alpha+\beta$ and $\alpha\beta$ as polynomials in $a_0^{1/p^n}, b_0^{1/p^n}, \cdots, a_n, b_n$, although for $n > 1$ these polynomials are too complicated for practical computations. We refer curious readers to the book of Serre [**Ser79**, §II.6] for details.

Our main objective for this subsection is to discuss fundamental theorems of Dieudonné and Manin which describe p-divisible groups over k via modules over $W(k)$ with a semilinear endomorphism. We won't provide their proofs, since we will only use these theorems as motivations for some constructions in Chapters III and IV. Curious readers may consult the book of Demazure [**Dem72**, Chapters III and IV] for an excellent exposition of these results.

Definition 2.3.7. Let us write σ for the Frobenius automorphism of $W(k)$.

(1) Given $W(k)$-modules M, N and an integer r, we say that an additive map $f : M \to N$ is σ^r-*semilinear* if it satisfies the identity

$$f(cm) = \sigma^r(c)f(m) \qquad \text{for each } c \in W(k) \text{ and } m \in M.$$

(2) A *Dieudonné module* over k is a free $W(k)$-module M with a σ-semilinear endomorphism φ_M, called the *Frobenius endomorphism* of M, whose image contains pM.

(3) A $W(k)$-linear map $f : M_1 \to M_2$ for Dieudonné modules M_1 and M_2 over k is a *morphism of Dieudonné modules* if it satisfies the identity $f \circ \varphi_{M_1} = \varphi_{M_2} \circ f$.

Lemma 2.3.8. The ring $W(k)$ is a complete discrete valuation ring with residue field k and uniformizer p.

Proof. Since $W(k)$ is p-adically complete with $W(k)/pW(k) \cong k$ by construction, it is a local ring with maximal ideal $pW(k)$ and residue field k by some general facts stated in the Stacks project [**Sta**, Tag 05GI and Tag 00E9]. Moreover, it is evident by Proposition 2.3.4 that every element $\alpha \in W(k)$ admits a unique expression $\alpha = p^n u$ for some integer $n \geq 0$ and unit $u \in W(k)$. Therefore we establish the desired assertion. □

Lemma 2.3.9. Let M be a Dieudonné module over k.

(1) The Frobenius endomorphism φ_M is injective.

(2) There exists a unique σ^{-1}-semilinear endomorphism ψ_M on M such that $\varphi_M \circ \psi_M$ and $\psi_M \circ \varphi_M$ coincide with the multiplication by p on M.

Proof. Take $e_1, \cdots, e_r \in M$ which form a basis over $W(k)$. Since $W(k)$ is a principal ideal domain by Lemma 2.3.8, statement (1) follows from the rank-nullity theorem and the fact that $\varphi_M(M)$ has rank r for containing pM. Hence we only need to prove statement (2).

We may write $pe_i = \varphi_M(e_i')$ for a unique element $e_i' \in M$ and in turn obtain a unique σ^{-1}-semilinear endomorphism ψ_M on M with $\varphi_M \circ \psi_M$ being the multiplication by p on M; indeed, ψ_M maps each e_i to e_i'. We wish to show that $\psi_M \circ \varphi_M$ coincides with the multiplication by p on M. Since we have $\psi_M(\varphi_M(e_i')) = \psi_M(\varphi_M(\psi_M(e_i))) = \psi_M(pe_i) = pe_i'$, we observe that $\psi_M \circ \varphi_M$ and the multiplication by p agree on the $W(k)$-module $M' \subseteq M$ spanned by $e_1', \cdots, e_r'$. Moreover, M' has rank r as $e_1', \cdots, e_r'$ are linearly independent by construction. Hence we deduce from the rank-nullity theorem that the difference between $\psi_M \circ \varphi_M$ and the multiplication by p identically vanishes on M, thereby establishing the desired assertion. □

Definition 2.3.10. Given a Dieudonné module M over k, we refer to the σ^{-1}-semilinear endomorphism ψ_M in Lemma 2.3.9 as the *Verschiebung endomorphism* of M.

Lemma 2.3.11. Given a Dieudonné module M over k, its dual $M^\vee = \mathrm{Hom}_{W(k)}(M, W(k))$ is naturally a Dieudonné module over k with

$$\varphi_{M^\vee}(f)(m) = \sigma(f(\psi_M(m))) \qquad \text{for all } f \in M^\vee \text{ and } m \in M.$$

Proof. The assertion is straightforward to verify by definition. □

THEOREM 2.3.12 (Dieudonné [**Die55**]). There is an exact anti-equivalence of categories

$$\mathbb{D} : \{ \, p\text{-divisible groups over } k \, \} \xrightarrow{\sim} \{ \, \text{Dieudonné modules over } k \, \}$$

such that for every p-divisible group G over k we have the following statements:

(1) The rank of $\mathbb{D}(G)$ is equal to the height of G.
(2) The maps φ_G, ψ_G, and $[p]_G$ yield $\varphi_{\mathbb{D}(G)}$, ψ_M, and the multiplication by p.
(3) There exists a natural isomorphism $\mathbb{D}(G^\vee) \cong \mathbb{D}(G)^\vee$.

Remark. Let us briefly describe the construction of $\mathbb{D}(G)$ for a p-divisible group $G = \varinjlim G_v$ over k. For each integer $n \geq 1$, we have a k-group W_n with $W_n(A) = W(A)/p^n W(A)$ for every perfect k-algebra A. If $G^\vee$ is connected, $\mathbb{D}(G) := \varprojlim_v \varinjlim_n \mathrm{Hom}_{k\text{-grp}}(G_v, W_n)$ turns out to be a Dieudonné module over k. with Frobenius endomorphism induced by φ_G. If $G^\vee$ is étale, it is connected by Theorem 2.2.23 and consequently yields a Dieudonné module $\mathbb{D}(G) := \mathbb{D}(G^\vee)^\vee$ over k. In the general case, G admits a natural decomposition

$$G \cong G^{\mathrm{unip}} \times G^{\mathrm{mult}}$$

with $(G^{\mathrm{unip}})^\vee$ connected and $(G^{\mathrm{mult}})^\vee$ étale, thereby giving rise to a Dieudonné module $\mathbb{D}(G) := \mathbb{D}(G^{\mathrm{unip}}) \oplus \mathbb{D}(G^{\mathrm{mult}})$ over k.

Definition 2.3.13. We refer to the functor $\mathbb{D}$ in Theorem 2.3.12 as the *Dieudonné functor*.

Example 2.3.14. We describe the Dieudonné functor for some simple p-divisible groups.

(1) $\mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p})$ is isomorphic to $W(k)$ with $\varphi_{\mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p})} = \sigma$ and $\psi_{\mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p})} = p\sigma^{-1}$.
(2) $\mathbb{D}(\mu_{p^\infty})$ is isomorphic to $W(k)$ with $\varphi_{\mathbb{D}(\mu_{p^\infty})} = p\sigma$ and $\psi_{\mathbb{D}(\mu_{p^\infty})} = \sigma^{-1}$.

Definition 2.3.15. Let us write $K_0(k) := W(k)[1/p]$ for the fraction field of $W(k)$.

(1) We define the *Frobenius automorphism* of $K_0(k)$ to be the unique field automorphism on $K_0(k)$ which extends σ.
(2) An *isocrystal* over $K_0(k)$ is a vector space N over $K_0(k)$ with a σ-semilinear automorphism φ_N called the *Frobenius automorphism* of N.
(3) A $K_0(k)$-linear map $g : N_1 \to N_2$ for isocrystals N_1 and N_2 over $K_0(k)$ is a *morphism of isocrystals* if it satisfies the identity
$$g(\varphi_{N_1}(n)) = \varphi_{N_2}(g(n)) \qquad \text{for each } n \in N_1.$$

LEMMA 2.3.16. Let σ denote the Frobenius automorphism of $K_0(k)$.

(1) Every Dieudonné module M over k yields an isocrystal $M[1/p] = M \otimes_{W(k)} K_0(k)$ over $K_0(k)$ with Frobenius automorphism $\varphi_M \otimes 1$.
(2) Given an isocrystal N over $K_0(k)$, its dual $N^\vee = \mathrm{Hom}_{K_0(k)}(N, K_0(k))$ is naturally an isocrystal over $K_0(k)$ with
$$\varphi_{N^\vee}(f)(n) = \sigma(f(\varphi_N^{-1}(n))) \qquad \text{for all } f \in N^\vee \text{ and } n \in N.$$
(3) Given two isocrystals N_1 and N_2 over $K_0(k)$, their tensor product $N_1 \otimes_{K_0(k)} N_2$ is naturally an isocrystal over $K_0(k)$ with Frobenius automorphism $\varphi_{N_1} \otimes \varphi_{N_2}$.

PROOF. All statements are straightforward to verify by definition. □

Example 2.3.17. For an isocrystal N of rank r over $K_0(k)$, its *determinant* $\det(N) := \wedge^r(N)$ is naturally an isocrystal of rank 1 over $K_0(k)$ as easily seen by Lemma 2.3.16.

Definition 2.3.18. We say that a homomorphism of group schemes or p-divisible groups is an *isogeny* if it is surjective with finite flat kernel.

Example 2.3.19. We present some examples of isogenies between p-divisible groups.

(1) Given a p-divisible group G over k, the maps $[p]_G, \varphi_G$, and ψ_G are all isogenies by Proposition 2.1.7 and Proposition 2.1.15.
(2) An isogeny $A \to B$ of two abelian varieties over k induces an isogeny $A[p^\infty] \to B[p^\infty]$.

Proposition 2.3.20. A homomorphism $f : G \to H$ of p-divisible groups over k is an isogeny if and only if it induces an isomorphism $\mathbb{D}(H)[1/p] \simeq \mathbb{D}(G)[1/p]$.

Proof. Let us first assume that f is an isogeny. Its kernel lies in G_v for some $v \geq 1$ and thus is a p-power torsion. Hence Theorem 2.3.12 implies that the map $\mathbb{D}(H) \to \mathbb{D}(G)$ induced by f is injective with its cokernel killed by a power of p. We deduce that f induces an isomorphism $\mathbb{D}(H)[1/p] \simeq \mathbb{D}(G)[1/p]$.

For the converse, we now assume that f induces an isomorphism $\mathbb{D}(H)[1/p] \simeq \mathbb{D}(G)[1/p]$. The map $\mathbb{D}(H) \to \mathbb{D}(G)$ is injective with $\mathbb{D}(H)$ and $\mathbb{D}(G)$ having the same rank over $W(k)$. Hence its cokernel is a p-power torsion by Lemma 2.3.8. Now we deduce from Theorem 2.3.12 that f is an isogeny as desired. □

Definition 2.3.21. Let N be an isocrystal over $K_0(k)$.

(1) The *degree* of N is the largest integer $\deg(N)$ with $\varphi_{\det(N)}(1) \in p^{\deg(N)}W(k)$, where we fix an isomorphism $\det(N) \simeq W(k)$.
(2) We write $\mathrm{rk}(N)$ for the rank of N and define the *slope* of N to be $\mu(N) := \dfrac{\deg(N)}{\mathrm{rk}(N)}$.

Example 2.3.22. Let $\lambda = d/r$ be a rational number written in lowest terms with $r > 0$. The *simple isocrystal of slope* λ over $K_0(k)$ is an isocrystal $N(\lambda)$ over $K_0(k)$ of rank r with

$$\varphi_{N(\lambda)}(e_1) = e_2, \cdots, \varphi_{N(\lambda)}(e_{r-1}) = e_r, \varphi_{N(\lambda)}(e_r) = p^d e_1,$$

where $e_1, \cdots, e_r$ are basis vectors. It is evident that $N(\lambda)$ has rank r, degree d, and slope λ.

Proposition 2.3.23. Given a p-divisible group G over k of height h and dimension d, the associated isocrystal $\mathbb{D}(G)[1/p]$ over $K_0(k)$ has rank h and degree d.

Proof. As noted in Proposition 2.2.22 and Example 2.3.19, the Frobenius φ_G is an isogeny with $\ker(\varphi_G)$ having order p^d. Moreover, Proposition 2.1.15 implies that $\ker(\varphi_G)$ is p-torsion. Hence we deduce from Theorem 2.3.12 and Lemma 2.3.8 that $\varphi_{\mathbb{D}(G)}$ is injective with $\mathrm{coker}(\varphi_{\mathbb{D}(G)}) \simeq (W(k)/pW(k))^{\oplus d}$. Now it is straightforward to verify that $\mathbb{D}(G)[1/p]$ has degree d. Since $\mathbb{D}(G)[1/p]$ evidently has rank h over $K_0(k)$ by Theorem 2.3.12, we establish the desired assertion. □

Theorem 2.3.24 (Manin [**Man63**]). Every isocrystal N over $K_0(\overline{k})$ admits a unique direct sum decomposition of the form

$$N \simeq \bigoplus_{i=1}^{l} N(\lambda_i)^{\oplus m_i} \quad \text{with } \lambda_1 < \lambda_2 < \cdots < \lambda_l.$$

Example 2.3.25. If an elliptic curve E over $\overline{\mathbb{F}}_p$ is ordinary, we have

$$\mathbb{D}(E[p^\infty])[1/p] \simeq N(0) \oplus N(1)$$

as easily seen by Example 2.2.25 and Example 2.3.14.

Remark. If E is supersingular, $\mathbb{D}(E[p^\infty])[1/p]$ turns out to be isomorphic to $N(1/2)$.

3. Hodge-Tate decomposition

In this section, we finally enter the realm of p-adic Hodge theory. Assuming some technical results, we prove the Hodge-Tate decomposition for Tate modules of p-divisible groups. The primary reference for this section is the article of Tate [**Tat67**].

3.1. Tate twists of p-adic representations

In this subsection, we introduce some basic notions in p-adic Hodge theory, such as p-adic fields, p-adic representations and their Tate twists. Given a valued field L, we write $\mathcal{O}_L$ for its valuation ring, $\mathfrak{m}_L$ for its maximal ideal, and k_L for its residue field.

Definition 3.1.1. A *p-adic field* is an extension of $\mathbb{Q}_p$ which is discretely valued and complete with a perfect residue field of characteristic p.

Example 3.1.2. We present some essential examples of p-adic fields.

(1) Every finite extension of $\mathbb{Q}_p$ is a p-adic field.
(2) Every perfect field k of characteristic p gives rises to a p-adic field $K_0(k) = W(k)[1/p]$ as noted in Lemma 2.3.8.

Remark. We will see in Chapter III, Proposition 2.2.18 that every p-adic field is a finite extension of $K_0(k)$ for some perfect field k of characteristic p.

For the rest of this section, we let K be a p-adic field with absolute Galois group Γ_K. We also write $\mathfrak{m}$ for its maximal ideal and k for its residue field.

Definition 3.1.3. A *p-adic representation* of Γ_K is a finite dimensional $\mathbb{Q}_p$-vector space V together with a continuous homomorphism $\Gamma_K \to \mathrm{GL}(V)$.

Example 3.1.4. Below are two important examples of p-adic representations.

(1) Given a p-divisible group G over K, its *rational Tate module* $V_p(G) := T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a p-adic Γ_K-representation by Proposition 2.1.17.
(2) For a K-variety X, the étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is a p-adic Γ_K-representation.

Definition 3.1.5. Given a $\mathbb{Z}_p[\Gamma_K]$-module M, its *n-th Tate twist* is the $\mathbb{Z}_p[\Gamma_K]$-module

$$M(n) := \begin{cases} M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes n} & \text{for } n \geq 0, \\ M \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p(1)^\vee)^{\otimes -n} & \text{for } n < 0 \end{cases}$$

where we set $\mathbb{Z}_p(1) := T_p(\mu_{p^\infty})$.

Example 3.1.6. The Galois group Γ_K acts on $\mathbb{Z}_p(1) = T_p(\mu_{p^\infty}) = \varprojlim \mu_{p^v}(\overline{K})$, via the homomorphism $\chi : \Gamma_K \to \mathrm{Aut}(\mathbb{Z}_p(1)) \cong \mathbb{Z}_p^\times$ called the *p-adic cyclotomic character* of K.

Lemma 3.1.7. Given a $\mathbb{Z}_p[\Gamma_K]$-module M, there exist natural Γ_K-equivariant isomorphisms

$$M(n) \cong M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n) \quad \text{and} \quad M(n)^\vee \cong M^\vee(-n) \quad \text{for each } n \in \mathbb{Z}.$$

Proof. The assertion is evident by definition. □

Lemma 3.1.8. If Γ_K acts on a $\mathbb{Z}_p$-module M via a homomorphism $\rho : \Gamma_K \to \mathrm{Aut}(M)$, it acts on $M(n)$ for each $n \in \mathbb{Z}$ via $\chi^n \cdot \rho$.

Proof. Under the identification $M(n) \cong M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n)$ given by Lemma 3.1.7, the Galois group Γ_K acts on $M(n)$ via $\rho \otimes \chi^n$. □

Definition 3.1.9. We define the *completed algebraic closure* of K to be $\mathbb{C}_K := \widehat{\overline{K}}$; in other words, $\mathbb{C}_K$ is the p-adic completion of the algebraic closure of K.

Remark. The field $\mathbb{C}_K$ is not a p-adic field as its valuation is not discrete.

Example 3.1.10. If K is an algebraic extension of $\mathbb{Q}_p$, we often write $\mathbb{C}_p = \mathbb{C}_K$ and refer to it as the *field of p-adic complex numbers.*

LEMMA 3.1.11. The action of Γ_K on $\overline{K}$ uniquely extends to a continuous action on $\mathbb{C}_K$.

PROOF. The assertion is obvious as the Γ_K-action on $\overline{K}$ is continuous. □

Definition 3.1.12. The *normalized p-adic valuation* on $\mathbb{C}_K$ is the unique valuation ν on $\mathbb{C}_K$ with $\nu(p) = 1$.

PROPOSITION 3.1.13. The field $\mathbb{C}_K$ is algebraically closed.

PROOF. We wish to prove that every nonconstant polynomial $f(t)$ over $\mathbb{C}_K$ admits a root in $\mathbb{C}_K$. Let us take an element $a \in \mathcal{O}_{\mathbb{C}_K}$ such that $af(t)$ is over $\mathcal{O}_{\mathbb{C}_K}$. If we denote the leading coefficient and the degree of $af(t)$ respectively by b and d, we have $ab^{d-1}f(t) = g(bt)$ for some monic polynomial $g(t)$ over $\mathcal{O}_{\mathbb{C}_K}$ of degree d. It suffices to show that $g(t)$ has a root in $\mathbb{C}_K$.

Let us write

$$g(t) = t^d + c_1 t^{d-1} + \cdots + c_d \quad \text{with } c_i \in \mathcal{O}_{\mathbb{C}_K}.$$

For each integer $n \geq 1$, we choose a polynomial

$$g_n(t) = t^d + c_{1,n} t^{d-1} + \cdots + c_{d,n}$$

with $c_{i,n} \in \mathcal{O}_{\overline{K}}$ and $\nu(c_i - c_{i,n}) \geq dn$. Since $\mathcal{O}_{\overline{K}}$ is integrally closed, each $g_n(t)$ admits a factorization into linear polynomials over $\mathcal{O}_{\overline{K}}$; in other words, we have

$$g_n(t) = \prod_{i=1}^{d} (t - \beta_{n,i}) \quad \text{with } \beta_{n,i} \in \mathcal{O}_{\overline{K}}. \tag{3.1}$$

Let us construct a sequence (α_n) in $\mathcal{O}_{\overline{K}}$ with $g_n(\alpha_n) = 0$ and $\nu(\alpha_n - \alpha_{n-1}) \geq n - 1$. We set $\alpha_1 := \beta_{1,1} \in \mathcal{O}_{\overline{K}}$ and proceed by induction on n. We have

$$g_n(\alpha_{n-1}) = g_n(\alpha_{n-1}) - g_{n-1}(\alpha_{n-1}) = \sum_{i=1}^{d} (c_{i,n} - c_{i,n-1}) \alpha_{n-1}^{d-i}$$

and thus find $\nu(g_n(\alpha_{n-1})) \geq d(n-1)$ as each $c_{i,n} - c_{i,n-1} = (c_{i,n} - c_i) + (c_i - c_{i,n-1})$ has valuation at least $d(n-1)$. We deduce from the identity (3.1) that $g_n(t)$ admits a root $\alpha_n = \beta_{n,i} \in \mathcal{O}_{\overline{K}}$ with $\nu(\alpha_{n-1} - \alpha_n) \geq n - 1$ and in turn obtain a desired sequece (α_n).

The sequence (α_n) is Cauchy by construction and thus converges to an element $\alpha \in \mathcal{O}_{\mathbb{C}_K}$. Moreover, for each integer $n \geq 1$ we have

$$g(\alpha_n) = g(\alpha_n) - g_n(\alpha_n) = \sum_{i=1}^{d} (c_i - c_{i,n}) \alpha_n^{d-i}$$

and consequently find $\nu(g(\alpha_n)) \geq dn$. Hence we deduce that α is a root of $g(t)$, thereby completing the proof. □

Remark. We can alternatively derive Proposition 3.1.13 from Krasner's lemma by modifying our argument. Moreover, we can use Krasner's lemma to show that $\overline{K}$ is not complete; in particular, we have $\mathbb{C}_K \neq \overline{K}$.

We assume the following fundamental result about the Tate twists of $\mathbb{C}_K$.

THEOREM 3.1.14 (Tate [**Tat67**], Sen [**Sen80**]). For the Galois cohomology of $\mathbb{C}_K$ and its Tate twists, we have the following statements:

(1) $H^0(\Gamma_K, \mathbb{C}_K)$ admits a natural isomorphism $H^0(\Gamma_K, \mathbb{C}_K) \cong K$.
(2) $H^1(\Gamma_K, \mathbb{C}_K)$ is an 1-dimensional vector space over K.
(3) $H^0(\Gamma_K, \mathbb{C}_K(n))$ and $H^1(\Gamma_K, \mathbb{C}_K(n))$ vanish for $n \neq 0$.

Remark. We refer curious readers to the notes of Brinon-Conrad [**BC**, §14] for a proof, which involves the higher ramification theory and the local class field theory.

LEMMA 3.1.15. Every p-adic Γ_K-representation V yields a natural $\mathbb{C}_K$-linear map

$$\tilde{\alpha}_V : \bigoplus_{n\in\mathbb{Z}} \left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K \mathbb{C}_K(n) \to V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$$

which is Γ_K-equivariant and injective.

PROOF. For each $n \in \mathbb{Z}$, we have a Γ_K-equivariant injective K-linear map

$$\tilde{\alpha}_{V,K}^{(n)} : \left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K K(n) \hookrightarrow V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n) \otimes_K K(n) \cong V \otimes_{\mathbb{Q}_p} \mathbb{C}_K.$$

Let us extend each $\tilde{\alpha}_{V,K}^{(n)}$ to a Γ_K-equivariant $\mathbb{C}_K$-linear map

$$\tilde{\alpha}_V^{(n)} : \left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K \mathbb{C}_K(n) \to V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$$

and take $\tilde{\alpha}_V := \bigoplus_{n\in\mathbb{Z}} \tilde{\alpha}_V^{(n)}$. We wish to show that $\tilde{\alpha}_V$ is injective.

Assume for contradiction that $\ker(\tilde{\alpha}_V)$ is not trivial. For every $n \in \mathbb{Z}$, we choose a basis $(v_{m,n})$ of $\left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K K(n)$ over K and regard each $v_{m,n}$ as a vector in $V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ via the map $\tilde{\alpha}_{V,K}^{(n)}$. Our assumption means that there exists a nontrivial linear relation $\sum c_{m,n} v_{m,n} = 0$ with minimum number of nonzero terms. Without loss of generality, we may set $c_{m_0,n_0} = 1$ for some integer m_0 and n_0. For every $\gamma \in \Gamma_K$, we find

$$0 = \gamma\left(\sum c_{m,n} v_{m,n}\right) - \chi(\gamma)^{n_0}\left(\sum c_{m,n} v_{m,n}\right) = \sum \left(\gamma(c_{m,n})\chi(\gamma)^n - \chi(\gamma)^{n_0} c_{m,n}\right) v_{m,n}$$

by Lemma 3.1.8 and the Γ_K-equivariance of $\tilde{\alpha}_V$. Since the coefficient of v_{m_0,n_0} in the last expression is 0, the minimality of our linear relation implies that all coefficients in the last expression must vanish and in turn yields the relation

$$\gamma(c_{m,n})\chi(\gamma)^{n-n_0} = c_{m,n} \qquad \text{for every } \gamma \in \Gamma_K.$$

Now Lemma 3.1.8 and Theorem 3.1.14 together imply that each $c_{m,n}$ lies in K with $c_{m,n} = 0$ for $n \neq n_0$. Hence we have a nontrivial K-linear relation $\sum c_{m,n_0} v_{m,n_0} = 0$ on the basis (v_{m,n_0}) of $\left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n_0)\right)^{\Gamma_K} \otimes_K K(n_0)$, thereby obtaining a desired contradiction. □

Definition 3.1.16. We say that a p-adic Γ_K-representation V is *Hodge-Tate* if the natural map $\tilde{\alpha}_V$ in Lemma 3.1.15 is an isomorphism.

Remark. We will see in §3.4 that p-adic representations presented in Example 3.1.4 are Hodge-Tate in many cases.

3.2. Points on p-divisible groups

For the rest of this section, we take the base ring to be $R = \mathcal{O}_K$. The main objective for this section is to investigate points on p-divisible groups over $\mathcal{O}_K$. We let L denote the p-adic completion of an algebraic extension of K. A primary example of such a field is $\mathbb{C}_K$.

LEMMA 3.2.1. The valuation ring $\mathcal{O}_L$ is $\mathfrak{m}$-adically complete; in other words, there exists a natural isomorphism

$$\mathcal{O}_L \cong \varprojlim \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L.$$

PROOF. The ideal $\mathfrak{m}$ contains p as the residue field $k = \mathcal{O}_K/\mathfrak{m}$ is of characteristic p. Since $\mathcal{O}_K$ is a discrete valuation ring, we deduce that the p-adic topology coincides with the $\mathfrak{m}$-adic topology and consequently establish the desired assertion by observing that $\mathcal{O}_L$ is p-adically complete. □

Definition 3.2.2. Given a p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$, we define its *group of $\mathcal{O}_L$-valued points* to be

$$G(\mathcal{O}_L) := \varprojlim_i \varinjlim_v G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L).$$

Remark. Readers should be aware that $G(\mathcal{O}_L)$ is in general not equal to $\varinjlim_v G_v(\mathcal{O}_L)$. This subtlety comes from the fact that we take points on G as a formal $\mathcal{O}_K$-group. In fact, if we write $G_v = \operatorname{Spec}(A_v)$ for each $v \geq 1$, we argue as in Lemma 2.2.19 to naturally identify G with a formal $\mathcal{O}_K$-group $\mathscr{G} = \operatorname{Spf}(\varprojlim A_v)$ and find $G(\mathcal{O}_L) \cong \mathscr{G}(\mathcal{O}_L)$.

Example 3.2.3. We describe the $\mathcal{O}_L$-valued points for some p-divisible groups of height 1.

(1) The p-power roots of unity μ_{p^∞} admits a natural isomorphism

$$\mu_{p^\infty}(\mathcal{O}_L) \cong 1 + \mathfrak{m}_L.$$

In fact, since $\mathfrak{m}_L$ contains p, we identify $\varinjlim_v \mu_{p^v}(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$ with the image of $1+\mathfrak{m}_L$ in $\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L$ and thus obtain the desired isomorphism by Lemma 3.2.1.

(2) The constant p-divisible group $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ admits a natural isomorphism

$$\underline{\mathbb{Q}_p/\mathbb{Z}_p}(\mathcal{O}_L) \cong \mathbb{Q}_p/\mathbb{Z}_p.$$

In fact, since $\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L$ is connected, we have $\underline{\mathbb{Z}/p^v\mathbb{Z}}(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \cong \mathbb{Z}/p^v\mathbb{Z}$ and thus obtain the desired isomorphism.

PROPOSITION 3.2.4. Given a p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$, the group $G(\mathcal{O}_L)$ is naturally a $\mathbb{Z}_p$-module such that its torsion part $G(\mathcal{O}_L)_{\text{tors}}$ admits a natural identification

$$G(\mathcal{O}_L)_{\text{tors}} \cong \varinjlim_v \varprojlim_i G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L).$$

PROOF. Proposition 2.1.7 shows that each $\varinjlim_v G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$ is a $\mathbb{Z}_p$-module and in turn implies that $G(\mathcal{O}_L)$ is also a $\mathbb{Z}_p$-module. Therefore $G(\mathcal{O}_L)_{\text{tors}}$ consists of p-power torsions. In addition, we observe by Proposition 2.1.7 that the p^v-torsion part of each $\varinjlim_v G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$ is $G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$. Since filtered colimits are exact in the category of abelian groups as stated in the Stacks project [**Sta**, Tag 04B0], we deduce that the p^v-torsion part of $G(\mathcal{O}_L)$ is $\varprojlim_i G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$. The desired assertion is now evident. □

PROPOSITION 3.2.5. Given a p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$ with $G_v = \operatorname{Spec}(A_v)$, there exists a canonical isomorphism

$$G(\mathcal{O}_L) \cong \operatorname{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\varprojlim A_v, \mathcal{O}_L).$$

PROOF. For every continuous $\mathcal{O}_K$-algebra homomorphism $f : \varprojlim A_v \to \mathcal{O}_L$, the induced map $f_i : \varprojlim A_v \to \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L$ for each $i \geq 1$ factors through a natural surjection $\varprojlim A_v \twoheadrightarrow A_{w_i}$ for some $w_i \geq 1$. Hence we have a canonical map

$$\operatorname{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\varprojlim A_v, \mathcal{O}_L) \longrightarrow \varprojlim_i \varinjlim_v \operatorname{Hom}_{\mathcal{O}_K}(A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$$

which sends each $f \in \operatorname{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\varprojlim A_v, \mathcal{O}_L)$ to $(f_i) \in \varprojlim_i \varinjlim_v \operatorname{Hom}_{\mathcal{O}_K}(A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$. It is not hard to see that this map is an isomorphism by Lemma 3.2.1. Now we obtain the desired isomorphism from the natural identification

$$G(\mathcal{O}_L) \cong \varprojlim_i \varinjlim_v \operatorname{Hom}_{\mathcal{O}_K}(A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L),$$

thereby completing the proof. □

Remark. Proposition 3.2.5 is equivalent to a canonical isomorphism $G(\mathcal{O}_L) \cong \mathscr{G}(\mathcal{O}_L)$ for the formal $\mathcal{O}_K$-group $\mathscr{G} = \operatorname{Spf}(\varprojlim A_v)$.

PROPOSITION 3.2.6. Let G be a p-divisible group over $\mathcal{O}_K$.

(1) If G is connected of dimension d, it admits a $\mathbb{Z}_p$-module isomorphism

$$G(\mathcal{O}_L) \simeq \operatorname{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathcal{O}_K[[t_1, \cdots, t_d]], \mathcal{O}_L)$$

where the multiplication by p on the target is induced by $[p]_{\mu_G}$.

(2) If G is étale, $G(\mathcal{O}_L)$ is torsion with a natural isomorphism $G(\mathcal{O}_L) \cong \varinjlim G_v(\mathcal{O}_L/\mathfrak{m}\mathcal{O}_L)$.

PROOF. Statement (1) is evident by Lemma 2.2.19 and Proposition 3.2.5. Let us now assume for statement (2) that G is étale. Each G_v is formally étale by a general fact stated in the Stacks project [**Sta**, Tag 02HM]; in particular, there exists a natural isomorphism $G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \cong G_v(\mathcal{O}_L/\mathfrak{m}^{i+1}\mathcal{O}_L)$ for each integer $i \geq 1$. Hence we find

$$G(\mathcal{O}_L) = \varprojlim_i \varinjlim_v G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \cong \varinjlim G_v(\mathcal{O}_L/\mathfrak{m}\mathcal{O}_L)$$

and in turn deduce from Proposition 2.1.7 that $G(\mathcal{O}_L)$ is a torsion group. □

Remark. If L is a finite extension of K, we have $\mathfrak{m}\mathcal{O}_L = \mathfrak{m}_L^j$ for some integer $j \geq 1$ and thus find $G^{\text{ét}}(\mathcal{O}_L) \cong \varinjlim G_v^{\text{ét}}(\mathcal{O}_L/\mathfrak{m}\mathcal{O}_L) \cong \varinjlim G_v^{\text{ét}}(\mathcal{O}_L/\mathfrak{m}_L) \cong \varinjlim G_v^{\text{ét}}(k_L)$ where the second isomorphism follows from the fact that each $G_v^{\text{ét}}$ is formally étale as noted in the proof.

LEMMA 3.2.7. An $\mathcal{O}_K$-algebra homomorphism $f : \mathcal{O}_K[[t_1, \cdots, t_n]] \to L$ is continuous if and only if each $f(t_i)$ lies in $\mathfrak{m}_L$.

PROOF. The map f is continuous if and only if there exists an integer v with $f(t_i^v) \in \mathfrak{m}_L$ for each $i = 1, \cdots, n$. Hence the assertion follows from the fact that $\mathcal{O}_K$ is reduced. □

Remark. Proposition 3.2.6 and Lemma 3.2.7 together show that every p-divisible group G over $\mathcal{O}_K$ of dimension d gives rise to an isomorphism $G^\circ(\mathcal{O}_L) \simeq \mathfrak{m}_L^{\oplus d}$ with group law on $\mathfrak{m}_L^d$ induced by μ_G. It turns out that the multiplication and the inverse on $\mathfrak{m}_L^{\oplus d}$ are analytic functions; in other words, $G^\circ(\mathcal{O}_L) \simeq \mathfrak{m}_L^{\oplus d}$ is a p-adic analytic group.

Proposition 3.2.8. Every p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$ yields an exact sequence

$$0 \longrightarrow G^\circ(\mathcal{O}_L) \longrightarrow G(\mathcal{O}_L) \longrightarrow G^{\text{ét}}(\mathcal{O}_L) \longrightarrow 0.$$

Proof. The sequence is left exact as limits and filtered colimits are left exact in the category of abelian groups. Hence we only need show that the map $G(\mathcal{O}_L) \to G^{\text{ét}}(\mathcal{O}_L)$ is surjective. For each integer $v \geq 1$, we let A_v, A_v°, and $A_v^{\text{ét}}$ respectively denote the affine rings of G_v, G_v°, and $G_v^{\text{ét}}$. In addition, we write $\mathscr{A} := \varprojlim A_v$, $\mathscr{A}^\circ := \varprojlim A_v^\circ$, and $\mathscr{A}^{\text{ét}} := \varprojlim A_v^{\text{ét}}$. By Proposition 3.2.5, it suffices to prove the surjectivity of the map

$$\mathrm{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathscr{A}, \mathcal{O}_L) \to \mathrm{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathscr{A}^{\text{ét}}, \mathcal{O}_L). \tag{3.2}$$

Lemma 2.2.19 yields a homeomorphic $\mathcal{O}_K$-algebra isomorphism

$$\mathscr{A}^\circ \simeq \mathcal{O}_K[[t_1, \cdots, t_d]]$$

where d denotes the dimension of G. Since k is perfect, we apply Proposition 1.4.15 to obtain a homeomorphic k-algebra isomorphism

$$(\mathscr{A}^{\text{ét}} \otimes_{\mathcal{O}_K} k)[[t_1, \cdots, t_d]] \simeq (\mathscr{A}^\circ \otimes_{\mathcal{O}_K} k)\widehat{\otimes}_k(\mathscr{A}^{\text{ét}} \otimes_{\mathcal{O}_K} k) \cong \mathscr{A} \otimes_{\mathcal{O}_K} k.$$

By Lemma 2.2.18, this map lifts to a surjective $\mathcal{O}_K$-algebra homomorphism

$$\theta : \mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]] \to \mathscr{A}.$$

Moreover, Lemma 2.2.18 shows that $\mathscr{A}$ is flat over $\mathcal{O}_K$ and in turn yields the relation $\ker(\theta) \otimes_{\mathcal{O}_K} k = 0$ by a general fact stated in the Stacks project [**Sta**, Tag 00HL]. For each $v \geq 1$, we take an ideal $\mathscr{J}_v$ of $\mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]]$ with $\mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]]/\mathscr{J}_v \cong A_v^\circ \otimes_{\mathcal{O}_K} A_v^{\text{ét}}$ and obtain a short exact sequence

$$0 \longrightarrow \ker(\theta)/\ker(\theta) \cap \mathscr{J}_v \longrightarrow \mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]]/\mathscr{J}_v \longrightarrow \mathscr{A}/\theta(\mathscr{J}_v) \longrightarrow 0.$$

We have $\mathfrak{m}\left(\ker(\theta)/\ker(\theta) \cap \mathscr{J}_v\right) = \ker(\theta)/\ker(\theta) \cap \mathscr{J}_v$ and thus find $\ker(\theta) = \ker(\theta) \cap \mathscr{J}_v$ for each $v \geq 1$ by Lemma 2.2.17 as $\mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]]/\mathscr{J}_v \cong A_v^\circ \otimes_{\mathcal{O}_K} A_v^{\text{ét}}$ is noetherian. Since we have $\bigcap \mathscr{J}_v = 0$, we see that $\ker(\theta)$ is trivial and in turn deduce that θ is an isomorphism.

The map θ is continuous as the kernel of each $\theta_v : \mathscr{A} \to A_v$ is open by the fact that the R-algebra A_v is of finite length. Moreover, with θ being a homeomorphism after base change to k we observe that every power of the ideal $\mathscr{I} := (t_1, \cdots, t_d)$ contains an open set in its image under θ and in turn find that θ is open. Hence θ is a homeomorphic R-algebra isomorphism. Now θ yields a surjective continuous map $\mathscr{A} \twoheadrightarrow \mathscr{A}^{\text{ét}}$ which splits the natural map $\mathscr{A}^{\text{ét}} \to \mathscr{A}$. We conclude that the map (3.2) is surjective as desired. □

Proposition 3.2.9. Let G be a p-divisible group over $\mathcal{O}_K$.

(1) For every $g \in G(\mathcal{O}_L)$, we have $p^n g \in G^\circ(\mathcal{O}_L)$ for each $n \gg 0$.
(2) If L is algebraically closed, $G(\mathcal{O}_L)$ is p-divisible in the sense that the multiplication by p on $G(\mathcal{O}_L)$ is surjective.

Proof. Since statement (1) is an immediate consequence of Proposition 3.2.6 and Proposition 3.2.8, we only need to establish statement (2). In light of Proposition 3.2.8, it suffices to show that the multiplication by p is surjective on each $G^{\text{ét}}(\mathcal{O}_L)$ and $G^\circ(\mathcal{O}_L)$. The surjectivity on $G^{\text{ét}}(\mathcal{O}_L)$ follows from Proposition 2.1.7 and Proposition 3.2.6. Moreover, we deduce the surjectivity on $G^\circ(\mathcal{O}_L)$ from Proposition 3.2.6 and the p-divisibility of μ_G. □

3.3. The logarithm for p-divisible groups

We continue to let L denote the p-adic completion of an algebraic extension of K. In this subsection, we construct and study the logarithm map for p-divisible groups over $\mathcal{O}_K$. For a p-divisible group G over $\mathcal{O}_K$ of dimension d, we work with a $\mathbb{Z}_p$-module isomorphism $G^\circ(\mathcal{O}_L) \simeq \mathrm{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathcal{O}_K[[t_1, \cdots, t_d]], \mathcal{O}_L)$ given by Proposition 3.2.6.

Definition 3.3.1. Let G be a p-divisible group over $\mathcal{O}_K$ and M be an $\mathcal{O}_K$-module. We write $\mathscr{I}$ for the augmentation ideal of μ_G.

(1) The *tangent space of* G *with values in* M is $t_G(M) := \mathrm{Hom}_{\mathcal{O}_K\text{-mod}}(\mathscr{I}/\mathscr{I}^2, M)$.
(2) The *cotangent space of* G *with values in* M is $t_G^*(M) := \mathscr{I}/\mathscr{I}^2 \otimes_{\mathcal{O}_K} M$.

Remark. We may naturally identify t_G and t_G^* respectively with the tangent space and the cotangent space of the formal group $\mathscr{G}\mu_G$ associated to μ_G. Our choice of a $\mathbb{Z}_p$-module isomorphism $G^\circ(\mathcal{O}_L) \simeq \mathrm{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathcal{O}_K[[t_1, \cdots, t_d]], \mathcal{O}_L)$ amounts to a choice of a formal $\mathcal{O}_K$-group isomorphism $\mathscr{G}\mu_G \simeq \mathrm{Spf}(\mathcal{O}_K[[t_1, \cdots, t_d]])$.

PROPOSITION 3.3.2. Given a p-divisible group G over $\mathcal{O}_K$ of dimension d, both $t_G(L)$ and $t_G^*(L)$ are vector spaces over L of dimension d.

PROOF. We identify the augmentation ideal of μ_G with $\mathscr{I} := (t_1, \cdots, t_d) \subseteq \mathcal{O}_K[[t_1, \cdots, t_d]]$ and obtain the assertion by observing that $\mathscr{I}/\mathscr{I}^2$ is a free $\mathcal{O}_K$-module of rank d. □

Definition 3.3.3. Given a p-divisible group G over $\mathcal{O}_K$, we define the *valuation filtration* on the group $G^\circ(\mathcal{O}_L)$ to be the collection $\left\{ \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) \right\}_{\lambda>0}$ with

$$\mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) := \{ f \in G^\circ(\mathcal{O}_L) : \nu(f(\alpha)) \geq \lambda \text{ for each } \alpha \in \mathscr{I} \}$$

where $\mathscr{I}$ denotes the augmentation ideal of μ_G.

Remark. We take λ to be a real number as the valuation on L may be nondiscrete.

LEMMA 3.3.4. Given a p-divisible group G over $\mathcal{O}_K$, we have

$$\bigcup_{\lambda>0} \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) = G^\circ(\mathcal{O}_L) \quad \text{and} \quad \bigcap_{\lambda>0} \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) = 0.$$

PROOF. The assertion is evident by Lemma 3.2.7 and the completeness of $\mathcal{O}_L$. □

LEMMA 3.3.5. Let G be a p-divisible group over $\mathcal{O}_K$ and λ be a positive real number. For every $f \in \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L)$, we have $pf \in \mathrm{Fil}^\kappa G^\circ(\mathcal{O}_L)$ with $\kappa = \min(\lambda + 1, 2\lambda)$.

PROOF. Let $\mathscr{I}$ denote the augmentation ideal of μ_G and take an arbitrary element $\alpha \in \mathscr{I}$. We may write $[p]_{\mu_G}(\alpha) = p\alpha + \beta$ for some $\beta \in \mathscr{I}^2$ by Lemma 2.2.13 and in turn find

$$(pf)(\alpha) = f([p]_{\mu_G}(\alpha)) = f(p\alpha + \beta) = pf(\alpha) + f(\beta).$$

Therefore we have $\nu((pf)(\alpha)) \geq \min(\lambda + 1, 2\lambda)$ as desired. □

LEMMA 3.3.6. Let G be a p-divisible group over $\mathcal{O}_K$. If L is a finite extension of K, we have

$$\bigcap_{n=1}^{\infty} p^n G^\circ(\mathcal{O}_L) = 0.$$

PROOF. Since the valuation on L is discrete, there exists a minimum positive valuation δ on $\mathcal{O}_L$ given by the valuation of the uniformizer. Hence we find $p^n G^\circ(\mathcal{O}_L) \subseteq \mathrm{Fil}^{n\delta} G^\circ(\mathcal{O}_L)$ for each $n \geq 1$ by Lemma 3.3.5 and in turn deduce the desired assertion from Lemma 3.3.4. □

LEMMA 3.3.7. Let G be a p-divisible group over $\mathcal{O}_K$ and write $\mathscr{I}$ for the augmentation ideal of μ_G. There exists a map

$$\log_G : G(\mathcal{O}_L) \to t_G(L)$$

such that for every $g \in G(\mathcal{O}_L)$ and $\alpha \in \mathscr{I}$ we have

$$\log_G(g)(\overline{\alpha}) = \lim_{n\to\infty} \frac{(p^n g)(\alpha)}{p^n}$$

where $\overline{\alpha}$ denotes the image of α in $\mathscr{I}/\mathscr{I}^2$.

PROOF. Let us take arbitrary elements $g \in G(\mathcal{O}_L)$ and $\alpha \in \mathscr{I}$. We have $p^n g \in G^\circ(\mathcal{O}_L)$ for each $n \gg 0$ as noted in Proposition 3.2.9. Therefore Lemma 3.3.5 implies that there exists $c \in \mathbb{R}$ with $p^n g \in \mathrm{Fil}^{n+c}\, G^\circ(\mathcal{O}_L)$ for each $n \gg 0$ and in turn yields the inequality

$$\nu\left(\frac{(p^n g)(\beta)}{p^n}\right) \geq 2(n+c) - n = n + 2c \quad \text{ for each } \beta \in \mathscr{I}^2. \tag{3.3}$$

Meanwhile, for each $n \gg 0$ we find

$$\frac{(p^{n+1}g)(\alpha)}{p^{n+1}} - \frac{(p^n g)(\alpha)}{p^n} = \frac{(p^n g)([p]_{\mu_G}(\alpha))}{p^{n+1}} - \frac{(p^n g)(\alpha)}{p^n} = \frac{(p^n g)([p]_{\mu_G}(\alpha) - p\alpha)}{p^{n+1}}.$$

Since we have $[p]_{\mu_G}(\alpha) - p\alpha \in \mathscr{I}^2$ by Lemma 2.2.13, we deduce from the inequality (3.3) that the sequence $\left(\dfrac{(p^n g)(\alpha)}{p^n}\right)$ converges in L. Moreover, if α lies in $\mathscr{I}^2$ the inequality (3.3) shows that the sequence converges to 0. The desired assertion is now evident. □

Definition 3.3.8. Given a p-divisible group G over $\mathcal{O}_K$, we refer to the map $\log_G$ given by Lemma 3.3.7 as the *logarithm* of G.

Example 3.3.9. Let us provide an explicit description of $\log_{\mu_{p^\infty}}$. Under the isomorphism $\mu_{p^\infty}(\mathcal{O}_L) \cong 1 + \mathfrak{m}_L$ noted in Example 3.2.3, each $g \in \mu_{p^\infty}(\mathcal{O}_L) \simeq \mathrm{Hom}_{\mathcal{O}_K\text{-cont}}(\mathcal{O}_L[[t]], \mathcal{O}_L)$ maps to $1 + g(t)$. In addition, $t_{\mu_{p^\infty}}$ admits an identification $t_{\mu_{p^\infty}}(L) \cong L$. Since we have $\mu_{\widehat{\mathbb{G}_m}}[p^\infty] \cong \mu_{p^\infty}$ as noted in Example 2.2.12, for each $g \in \mu_{p^\infty}(\mathcal{O}_L)$ we find

$$(p^n g)(t) = g\left((1+t)^{p^n} - 1\right) = (1 + g(t))^{p^n} - 1$$

and thus obtain the identity

$$\log_{\mu_{p^\infty}}(1+x) = \lim_{n\to\infty} \frac{(1+x)^{p^n} - 1}{p^n} = \lim_{n\to\infty} \sum_{i=1}^{p^n} \frac{1}{p^n}\binom{p^n}{i} x^i \quad \text{ for each } x \in \mathfrak{m}_L.$$

Moreover, for integers i and n we have

$$\frac{1}{p^n}\binom{p^n}{i} - \frac{(-1)^{i-1}}{i} = \frac{(p^n-1)\cdots(p^n-i+1) - (-1)^{i-1}(i-1)!}{i!}.$$

We observe that the numerator is divisible by p^n and in turn find

$$\nu\left(\frac{1}{p^n}\binom{p^n}{i} - \frac{(-1)^{i-1}}{i}\right) \geq n - \nu(i!) \geq n - \sum_{j=1}^{\infty} \frac{i}{p^j} = n - \frac{i}{p-1}.$$

Hence we obtain the expression

$$\log_{\mu_{p^\infty}}(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} x^i \quad \text{ for each } x \in \mathfrak{m}_L,$$

which coincides with the p-adic logarithm.

Let us state the following technical result about the logarithm maps without a proof.

PROPOSITION 3.3.10. Given a p-divisible group G over $\mathcal{O}_K$, the map $\log_G$ is a local homeomorphism in the sense that it induces an isomorphism

$$\mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) \simeq \left\{ \tau \in t_G(L) : \nu(\tau(f)) \geq \lambda \text{ for each } f \in \mathscr{I}/\mathscr{I}^2 \right\} \qquad \text{for every } \lambda \geq 1.$$

Remark. A key fact for the proof of Proposition 3.3.10 is that the multiplication by p on the group $G^\circ(\mathcal{O}_L)$ induces an isomorphism $\mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) \cong \mathrm{Fil}^{\lambda+1} G^\circ(\mathcal{O}_L)$ as stated in the book of Serre [**Ser92**, Theorem 9.4]. It turns out that $\log_G$ admits a local inverse $\exp_G^\lambda$ on

$$\mathrm{Fil}^\lambda t_G(L) := \left\{ \tau \in t_G(L) : \nu(\tau(f)) \geq \lambda \text{ for each } f \in \mathscr{I}/\mathscr{I}^2 \right\}.$$

In fact, for every $\tau \in \mathrm{Fil}^\lambda t_G(L)$ we have $\exp_G^\lambda(\tau)(t_i) = \lim\limits_{n\to\infty} g_n(t_i)$ with each $g_n \in \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L)$ determined by the relation $(p^n g_n)(t_i) = p^n \tau(t_i)$.

PROPOSITION 3.3.11. Let G be a p-divisible group over $\mathcal{O}_K$ and denote by $\mathscr{I}$ the augmentation ideal of μ_G.

(1) $\log_G$ is a group homomorphism.
(2) The kernel of $\log_G$ is the torsion subgroup $G(\mathcal{O}_L)_{\mathrm{tors}}$ of $G(\mathcal{O}_L)$.
(3) $\log_G$ induces an isomorphism $G(\mathcal{O}_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq t_G(L)$.

PROOF. Let us write $\mathscr{A}^\circ := \mathcal{O}_K[[t_1, \cdots, t_d]]$ where d is the dimension of G. Take arbitrary elements $g,\ h \in G(\mathcal{O}_L)$ and $\alpha \in \mathscr{I}$. We have $p^n g,\ p^n h \in G^\circ(\mathcal{O}_L)$ for each $n \gg 0$ as noted in Proposition 3.2.9. Since the axioms for μ_G yield the relation

$$\mu_G(\alpha) \in 1 \otimes \alpha + \alpha \otimes 1 + (\mathscr{I} \widehat{\otimes}_{\mathscr{A}^\circ} \mathscr{I})^2,$$

for each $n \gg 0$ we may write

$$(p^n(g+h))(\alpha) = (p^n g \otimes p^n h) \circ \mu_G(\alpha) = (p^n g)(\alpha) + (p^n h)(\alpha) + \beta_n$$

with $\beta_n \in (p^n g)(\mathscr{I}) \cdot (p^n h)(\mathscr{I})$. Moreover, we deduce from Lemma 3.3.5 that there exists $c \in \mathbb{R}$ with $p^n g,\ p^n h \in \mathrm{Fil}^{n+c} G^\circ(\mathcal{O}_L)$ for each $n \gg 0$ and in turn find $\nu(\beta_n) \geq 2(n+c)$. Now we obtain the identity

$$\lim_{n\to\infty} \frac{(p^n(g+h))(\alpha)}{p^n} = \lim_{n\to\infty} \frac{(p^n g)(\alpha)}{p^n} + \lim_{n\to\infty} \frac{(p^n h)(\alpha)}{p^n}$$

and consequently establish statement (1).

For statement (2), we only need to show that $\ker(\log_G)$ lies in $G(\mathcal{O}_L)_{\mathrm{tors}}$; indeed, we have $G(\mathcal{O}_L)_{\mathrm{tors}} \subseteq \ker(\log_G)$ by the fact that $t_G(L)$ is torsion free for being a vector space over L. Let us take an arbitrary element $g \in \ker(\log_G)$. Proposition 3.2.9 and Lemma 3.3.5 together imply that we have $p^n g \in \mathrm{Fil}^1 G^\circ(\mathcal{O}_L)$ for some $n \gg 0$. Since $p^n g$ lies in $\ker(\log_G)$ by statement (1), it must vanish by Proposition 3.3.10. We deduce that g is a torsion element and thus obtain statement (2).

Statement (2) readily implies that $\log_G$ induces an injective map $G(\mathcal{O}_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to t_G(L)$. Moreover, we observe by Proposition 3.3.10 that this map is also surjective as for each $\tau \in t_G(L)$ there exists an integer n with $p^n \tau \in \mathrm{Fil}^1 t_G(L)$. Hence we establish statement (3), thereby completing the proof. $\square$

3.4. Hodge-Tate decomposition for the Tate module

In this subsection, we establish the first main result for this chapter by exploiting our accumulated knowledge of finite flat group schemes and p-divisible groups.

Lemma 3.4.1. Every p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$ yields canonical isomorphisms

$$G_v(\overline{K}) \cong G_v(\mathbb{C}_K) \cong G_v(\mathcal{O}_{\mathbb{C}_K}) \quad \text{for each } v \geq 1.$$

Proof. Since the generic fiber of each G_v is finite étale as easily seen by Corollary 1.3.11, the first isomorphism follows from Proposition 3.1.13 and a standard fact stated in the Stacks project [**Sta**, Tag 0BND]. The second isomorphism is evident by the valuative criterion. □

Lemma 3.4.2. For every p-divisible group G over $\mathcal{O}_K$, we have natural identifications

$$G(\mathcal{O}_{\mathbb{C}_K})^{\Gamma_K} \cong G(\mathcal{O}_K) \quad \text{and} \quad t_G(\mathbb{C}_K)^{\Gamma_K} \cong t_G(K).$$

Proof. Theorem 3.1.14 yields canonical identifications $\mathbb{C}_K^{\Gamma_K} = K$ and $\mathcal{O}_{\mathbb{C}_K}^{\Gamma_K} = \mathcal{O}_K$. Hence the desired isomorphisms follow from Proposition 3.2.5 and Definition 3.3.1. □

Definition 3.4.3. Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$.

(1) The *Tate module* of G is $T_p(G) := T_p(G \times_{\mathcal{O}_K} K) = \varprojlim G_v(\overline{K})$.

(2) The *Tate comodule* of G is $\Phi_p(G) := \varinjlim G_v(\overline{K})$.

Example 3.4.4. We have $T_p(\mu_{p^\infty}) = \mathbb{Z}_p(1)$ by definition and identify $\Phi_p(\mu_{p^\infty}) = \varinjlim \mu_{p^v}(\overline{K})$ with the group of p-power torsions in $\overline{K}$.

Lemma 3.4.5. Given a p-divisible group $G = \varinjlim G_v$ of height h over $\mathcal{O}_K$, its Tate module $T_p(G)$ is a free $\mathbb{Z}_p$-module of rank h.

Proof. We note by Corollary 1.3.11 that the generic fiber of each G_v is finite étale and in turn deduce from Proposition 1.3.4 that $G_v(\overline{K})$ is a finite abelian group of order p^{vh}. Hence the desired assertion follows from Proposition 2.1.17. □

Remark. We can also show that $\Phi_p(G)$ is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus h}$.

Lemma 3.4.6. Every p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$ gives rise to a natural surjective $\mathbb{Z}_p$-module homomorphism $T_p(G^\vee) \twoheadrightarrow T_p((G^\circ)^\vee)$.

Proof. For each $v \geq 1$, Proposition 1.2.13 and Lemma 2.1.6 yield a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (G_{v+1}^{\text{ét}})^\vee(\overline{K}) & \longrightarrow & G_{v+1}^\vee(\overline{K}) & \xrightarrow{\pi_{v+1}} & (G_{v+1}^\circ)^\vee(\overline{K}) & \longrightarrow & 0 \\ & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow{\scriptstyle j_v} & & \downarrow\!\!\!\downarrow & & \\ 0 & \longrightarrow & (G_v^{\text{ét}})^\vee(\overline{K}) & \longrightarrow & G_v^\vee(\overline{K}) & \xrightarrow{\pi_v} & (G_v^\circ)^\vee(\overline{K}) & \longrightarrow & 0 \end{array}$$

where both rows are exact. We wish to show that for every $(w_v) \in \varprojlim (G_v^\circ)^\vee(\overline{K}) = T_p((G^\circ)^\vee)$ there exists an element $(\widetilde{w}_v) \in \varprojlim G_v^\vee(\overline{K}) = T_p(G^\vee)$ with $\pi_v(\widetilde{w}_v) = w_v$. Let us choose $\widetilde{w}_1 \in G_1^\vee(\overline{K})$ with $\pi_v(\widetilde{w}_v) = w_v$ and inductively construct $(\widetilde{w}_v)$. If we take an element $\widetilde{w}'_{v+1} \in G_{v+1}^\vee(\overline{K})$ with $\pi_{v+1}(\widetilde{w}'_{v+1}) = w_{v+1}$, we have $\pi_v(j_v(\widetilde{w}'_{v+1})) = w_v = \pi_v(\widetilde{w}_v)$ and thus find $j_v(\widetilde{w}'_{v+1}) = \widetilde{w}_v w''_v$ for some $w''_v \in (G_v^{\text{ét}})^\vee(\overline{K})$. Now we pick $w''_{v+1} \in (G_{v+1}^{\text{ét}})^\vee(\overline{K})$ with $j_v(w''_{v+1}) = w''_v$ and set $\widetilde{w}_{v+1} := \widetilde{w}'_{v+1}(w''_{v+1})^{-1}$ to deduce the desired assertion. □

Remark. We can alternatively deduce Lemma 3.4.6 from Proposition 1.2.13, Lemma 2.1.6, and a general fact stated in the Stacks project [**Sta**, Tag 0598].

PROPOSITION 3.4.7. Given a p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$, there exist canonical Γ_K-equivariant $\mathbb{Z}_p$-module isomorphisms

$$T_p(G) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1)) \quad \text{and} \quad \Phi_p(G) \cong \mathrm{Hom}_{\mathbb{Z}_p}\left(T_p(G^\vee), \Phi_p(\mu_{p^\infty})\right).$$

PROOF. Corollary 1.3.11 implies that the generic fiber of each G_v is finite étale. Hence each G_v gives rise to a canonical identification

$$G_v(\overline{K}) \cong (G_v^\vee)^\vee(\overline{K}) = \mathrm{Hom}_{\overline{K}\text{-grp}}\left((G_v^\vee)_{\overline{K}}, (\mu_{p^v})_{\overline{K}}\right) \cong \mathrm{Hom}(G_v^\vee(\overline{K}), \mu_{p^v}(\overline{K})) \tag{3.4}$$

by Theorem 1.2.4, Lemma 1.2.3, and Proposition 1.3.4. We deduce that $T_p(G)$ admits a natural Γ_K-equivariant isomorphism

$$\begin{aligned} T_p(G) &= \varprojlim G_v(\overline{K}) \cong \varprojlim \mathrm{Hom}(G_v^\vee(\overline{K}), \mu_{p^v}(\overline{K})) \\ &= \mathrm{Hom}_{\mathbb{Z}_p}(\varprojlim G_v^\vee(\overline{K}), \varprojlim \mu_{p^v}(\overline{K})) = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1)). \end{aligned}$$

Moreover, under the isomorphism $\Phi_p(G) = \varinjlim G_v(\overline{K}) \cong \varinjlim \mathrm{Hom}_{\mathbb{Z}_p}(G_v^\vee(\overline{K}), \Phi_p(\mu_{p^\infty}))$ given by the identification (3.4), we have a natural Γ_K-equivariant map

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \Phi_p(\mu_{p^\infty})) = \mathrm{Hom}_{\mathbb{Z}_p}(\varprojlim G_v^\vee(\overline{K}), \Phi_p(\mu_{p^\infty})) \longrightarrow \Phi_p(G)$$

which we verify to be an isomorphism using Lemma 2.1.6. □

PROPOSITION 3.4.8. Every p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$ yields a short exact sequence

$$0 \longrightarrow \Phi_p(G) \longrightarrow G(\mathcal{O}_{\mathbb{C}_K}) \xrightarrow{\log_G} t_G(\mathbb{C}_K) \longrightarrow 0.$$

PROOF. Since $G(\mathcal{O}_{\mathbb{C}_K})$ is p-divisible by Proposition 3.1.13 and Proposition 3.2.9, we deduce from Proposition 3.3.11 that $\log_G$ is surjective. In addition, we have

$$\ker(\log_G) = G(\mathcal{O}_{\mathbb{C}_K})_{\mathrm{tors}} \cong \varinjlim_v \varprojlim_i G_v(\mathcal{O}_{\mathbb{C}_K}/\mathfrak{m}^i\mathcal{O}_{\mathbb{C}_K}) = \varinjlim_v G_v(\mathcal{O}_{\mathbb{C}_K}) \cong \varinjlim_v G_v(\overline{K}) = \Phi_p(G)$$

by Proposition 3.3.11, Proposition 3.2.4, Lemma 3.2.1, and Lemma 3.4.1. □

LEMMA 3.4.9. Every p-divisible group G over $\mathcal{O}_K$ yields Γ_K-equivariant $\mathbb{Z}_p$-module maps

$$\alpha : G(\mathcal{O}_{\mathbb{C}_K}) \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}) \quad \text{and} \quad d\alpha : t_G(\mathbb{C}_K) \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$$

via a natural isomorphism $T_p(G^\vee) \cong \mathrm{Hom}_{p\text{-div grp}}\left(G_{\mathcal{O}_{\mathbb{C}_K}}, (\mu_{p^\infty})_{\mathcal{O}_{\mathbb{C}_K}}\right)$.

PROOF. Let us write $G = \varinjlim G_v$ where each G_v is a finite flat $\mathcal{O}_K$-group. Lemma 3.4.1 and Lemma 1.2.3 together yield a canonical identification

$$\begin{aligned} T_p(G^\vee) &= \varprojlim G_v^\vee(\overline{K}) \cong \varprojlim G_v^\vee(\mathcal{O}_{\mathbb{C}_K}) \\ &= \varprojlim \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_K}\text{-grp}}\left((G_v)_{\mathcal{O}_{\mathbb{C}_K}}, (\mu_{p^v})_{\mathcal{O}_{\mathbb{C}_K}}\right) \\ &= \mathrm{Hom}_{p\text{-div grp}}\left(G_{\mathcal{O}_{\mathbb{C}_K}}, (\mu_{p^\infty})_{\mathcal{O}_{\mathbb{C}_K}}\right). \end{aligned} \tag{3.5}$$

In addition, we have $\mu_{p^\infty}(\mathcal{O}_{\mathbb{C}_K}) \cong 1 + \mathfrak{m}_{\mathbb{C}_K}$ and $t_{\mu_{p^\infty}}(\mathbb{C}_K) \cong \mathbb{C}_K$ as noted in Example 3.3.9. Hence each $w \in T_p(G^\vee)$ gives rise to maps

$$w_{\mathcal{O}_{\mathbb{C}_K}} : G(\mathcal{O}_{\mathbb{C}_K}) \to \mu_{p^\infty}(\mathcal{O}_{\mathbb{C}_K}) \cong 1 + \mathfrak{m}_{\mathbb{C}_K} \quad \text{and} \quad dw_{\mathbb{C}_K} : t_G(\mathbb{C}_K) \to t_{\mu_{p^\infty}}(\mathbb{C}_K) \cong \mathbb{C}_K.$$

Now we obtain the desired maps α and $d\alpha$ by setting

$$\alpha(g)(w) := w_{\mathcal{O}_{\mathbb{C}_K}}(g) \quad \text{and} \quad d\alpha(\tau)(w) := dw_{\mathbb{C}_K}(\tau)$$

for each $g \in G(\mathcal{O}_{\mathbb{C}_K})$, $\tau \in t_G(\mathbb{C}_K)$, and $w \in T_p(G^\vee)$. □

PROPOSITION 3.4.10. Every p-divisible group G over $\mathcal{O}_K$ gives rise to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi_p(G) & \longrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\log_G} & t_G(\mathbb{C}_K) & \longrightarrow & 0 \\ & & \downarrow\wr & & \int\downarrow \alpha & & \int\downarrow d\alpha & & \\ 0 & \to & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \Phi_p(\mu_{p^\infty})) & \to & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K}) & \to & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) & \to & 0 \end{array}$$

with exact rows and Γ_K-equivariant vertical arrows.

PROOF. Let us first describe the maps in the diagram. The top row comes from Proposition 3.4.8 and is evidently exact. In addition, since we have $\mu_{p^\infty}(\mathcal{O}_{\mathbb{C}_K}) \cong 1 + \mathfrak{m}_{\mathbb{C}_K}$ and $t_{\mu_{p^\infty}}(\mathbb{C}_K) \cong \mathbb{C}_K$ as noted in Example 3.3.9, we obtain the bottom row by Proposition 3.4.8 and deduce its exactness as $T_p(G^\vee)$ is free over $\mathbb{Z}_p$ by Proposition 2.1.17. The vertical arrows are the natural Γ_K-equivariant maps given by Proposition 3.4.7 and Lemma 3.4.9.

It is straightforward to verify that the diagram is commutative. Hence it remains to prove that α and $d\alpha$ are injective. Since we have $\ker(\alpha) \simeq \ker(d\alpha)$ by the snake lemma, it suffices to show that $d\alpha$ is injective.

We assert that α is injective on $G(\mathcal{O}_K)$. Suppose for contradiction that there exists a nonzero element $g \in \ker(\alpha)$. The $\mathbb{Z}_p$-linear map $d\alpha$ is indeed $\mathbb{Q}_p$-linear as both $t_G(\mathbb{C}_K)$ and $\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$ are vector spaces over $\mathbb{Q}_p$. We deduce that $\ker(\alpha) \simeq \ker(d\alpha)$ is also a vector space over $\mathbb{Q}_p$ and thus is torsion free. Now we may assume by Proposition 3.2.9 that g lies in $G^\circ(\mathcal{O}_K)$. Lemma 3.4.9 yields a commutative diagram

$$\begin{array}{ccc} G^\circ(\mathcal{O}_{\mathbb{C}_K}) & \hookrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) \\ \downarrow \alpha^\circ & & \downarrow \alpha \\ \mathrm{Hom}_{\mathbb{Z}_p}(T_p((G^\circ)^\vee), 1+\mathfrak{m}_{\mathbb{C}_K}) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K}) \end{array}$$

where the injectivity of the horizontal maps follow from Proposition 3.2.8 and Lemma 3.4.6. Therefore we have $g \in \ker(\alpha^\circ) \cap G^\circ(\mathcal{O}_K)$ and also find $\ker(\alpha^\circ) \cap G^\circ(\mathcal{O}_K) = \ker(\alpha^\circ)^{\Gamma_K}$ by Lemma 3.4.2. Since $\ker(\alpha^\circ)^{\Gamma_K}$ is a vector space over $\mathbb{Q}_p$, for every integer $n \geq 0$ there exists an element $g_n \in \ker(\alpha^\circ) \cap G^\circ(\mathcal{O}_K)$ with $g = p^n g_n$. We deduce from Lemma 3.3.6 that g must be zero and in turn obtain a desired contradiction.

Now we show that $d\alpha$ is injective on $t_G(K)$. It is enough to establish the injectivity on $\log_G(G(\mathcal{O}_K))$ as we have $\log_G(G(\mathcal{O}_K)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = t_G(K)$ by Proposition 3.3.11. Let us take an arbitrary element $h \in G(\mathcal{O}_K)$ with $\log_G(h) \in \ker(d\alpha)$. Since $\log_G$ induces the isomorphism $\ker(\alpha) \simeq \ker(d\alpha)$ by the snake lemma, we find $\log_G(h) = \log_G(h')$ for some $h' \in \ker(\alpha)$. Proposition 3.3.11 implies that $h - h'$ is torsion, which means that there exists $n \geq 0$ with $p^n(h - h') = 0$ or equivalently $p^n h = p^n h'$. Hence we have $p^n h \in \ker(\alpha) \cap G(\mathcal{O}_K)$ and in turn find $p^n h = 0$ by the injectivity of α on $G(\mathcal{O}_K)$. We deduce from Proposition 3.3.11 that $\log_G(h)$ is zero, which implies that $d\alpha$ is injective on $\log_G(G(\mathcal{O}_K))$.

Our discussion in the previous paragraph shows that $d\alpha$ factors through an injective map

$$t_G(\mathbb{C}_K) \cong t_G(K) \otimes_K \mathbb{C}_K \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K} \otimes_K \mathbb{C}_K.$$

In addition, Lemma 3.1.15 yields an injective map

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K} \otimes_K \mathbb{C}_K \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), K) \otimes_K \mathbb{C}_K \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$$

where the isomorphism comes from the fact that $T_p(G^\vee)$ is free over $\mathbb{Z}_p$ by Lemma 3.4.5. Now we identify $d\alpha$ with the composition of these maps and in turn establish its injectivity, thereby completing the proof. □

THEOREM 3.4.11 (Tate [**Tat67**]). Let G be a p-divisible group over $\mathcal{O}_K$.

(1) There exist natural isomorphisms
$$G(\mathcal{O}_K) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K})^{\Gamma_K} \quad \text{and} \quad t_G(K) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K}.$$

(2) The tangent spaces $t_G(\mathbb{C}_K)$ and $t_{G^\vee}(\mathbb{C}_K)$ are orthogonal complements with respect to a $\mathbb{C}_K$-linear Γ_K-equivariant perfect pairing
$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \times \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) \to \mathbb{C}_K(-1).$$

PROOF. Proposition 3.4.10 and the snake lemma together yield a commutative diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K}) & \longrightarrow & \mathrm{coker}(\alpha) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \log_G} & & \big\downarrow & & \big\downarrow{\scriptstyle \wr} & & \\ 0 & \longrightarrow & t_G(\mathbb{C}_K) & \xrightarrow{d\alpha} & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) & \longrightarrow & \mathrm{coker}(d\alpha) & \longrightarrow & 0 \end{array}$$
where both rows are exact. We apply Lemma 3.4.2 to obtain a commutative diagram
$$\begin{array}{ccccccc} 0 & \longrightarrow & G(\mathcal{O}_K) & \xrightarrow{\alpha_K} & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K})^{\Gamma_K} & \longrightarrow & \mathrm{coker}(\alpha)^{\Gamma_K} \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \wr} \\ 0 & \longrightarrow & t_G(K) & \xrightarrow{d\alpha_K} & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K} & \longrightarrow & \mathrm{coker}(d\alpha)^{\Gamma_K} \end{array}$$
where both rows are exact. We observe that the middle vertical map induces an injective map
$$\mathrm{coker}(\alpha_K) \hookrightarrow \mathrm{coker}(d\alpha_K). \tag{3.6}$$
In addition, we switch the roles of G and $G^\vee$ to get an injective map
$$d\alpha_K^\vee : t_{G^\vee}(K) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K)^{\Gamma_K}.$$

Let us denote the height of G by h. Proposition 2.1.8 and Lemma 3.4.5 together show that $V := \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K)$ and $W := \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$ are vector spaces over $\mathbb{C}_K$ of dimension h. Moreover, Proposition 3.4.7 yields a Γ_K-equivariant $\mathbb{Z}_p$-linear perfect pairing
$$T_p(G) \times T_p(G^\vee) \to \mathbb{Z}_p(1),$$
which in turn gives rise to a Γ_K-equivariant $\mathbb{C}_K$-linear perfect pairing
$$V \times W \to \mathbb{C}_K(-1). \tag{3.7}$$
This pairing maps $V^{\Gamma_K} \times W^{\Gamma_K}$ into $\mathbb{C}_K(-1)^{\Gamma_K}$, which is zero by Theorem 3.1.14. We deduce that $V^{\Gamma_K} \otimes_K \mathbb{C}_K$ and $W^{\Gamma_K} \otimes_K \mathbb{C}_K$ are orthogonal and consequently find
$$\dim_K(V^{\Gamma_K}) + \dim_K(W^{\Gamma_K}) \leq \dim_{\mathbb{C}_K}(V) = h.$$
Meanwhile, the injectivity of $d\alpha_K$ and $d\alpha_K^\vee$ yields the inequality
$$\dim_K(V^{\Gamma_K}) + \dim_K(W^{\Gamma_K}) \geq \dim_K(t_G(K)) + \dim_K(t_{G^\vee}(K)) = h$$
where the equality follows from Theorem 2.2.23 and Proposition 3.3.2. Therefore all inequalities are in fact equalities. We deduce that the injective map $d\alpha_K$ is an isomorphism and in turn find by the injective map (3.6) that α_K is also an isomorphism. Now we establish statement (1), which in particular yields natural identifications
$$t_G(\mathbb{C}_K) \cong W^{\Gamma_K} \otimes_K \mathbb{C}_K \quad \text{and} \quad t_{G^\vee}(\mathbb{C}_K) \cong V^{\Gamma_K} \otimes_K \mathbb{C}_K.$$
Our discussion readily shows that these spaces are orthogonal under the pairing (3.7) with $\dim_{\mathbb{C}_K}(t_G(\mathbb{C}_K)) + \dim_{\mathbb{C}_K}(t_{G^\vee}(\mathbb{C}_K)) = \dim_{\mathbb{C}_K}(V)$, thereby implying statement (2). □

PROPOSITION 3.4.12. Given a p-divisible group G of dimension d over $\mathcal{O}_K$, we have

$$d = \dim_K \left(\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K}\right) = \dim_K(T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_K(-1))^{\Gamma_K}.$$

PROOF. The first equality is evident by Proposition 3.3.2 and Theorem 3.4.11. The second equality follows from the identification

$$T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_K(-1) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{C}_K(-1) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$$

given by Lemma 3.4.5 and Proposition 3.4.7. □

Remark. Lemma 3.4.5 and Proposition 3.4.12 together show that we can compute the height and the dimension of G from $T_p(G)$.

THEOREM 3.4.13 (Tate [**Tat67**]). Every p-divisible group G over $\mathcal{O}_K$ gives rise to a canonical $\mathbb{C}_K[\Gamma_K]$-module isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \cong t_{G^\vee}(\mathbb{C}_K) \oplus t_G^*(\mathbb{C}_K)(-1).$$

PROOF. We identify $t_G^*(\mathbb{C}_K)$ with the $\mathbb{C}_K$-dual $t_G(\mathbb{C}_K)$ and find

$$\mathrm{Hom}_{\mathbb{C}_K}(t_G(\mathbb{C}_K), \mathbb{C}_K(-1)) \cong t_G^*(\mathbb{C}_K)(-1).$$

Since Theorem 3.4.11 yields a $\mathbb{C}_K$-linear Γ_K-equivariant perfect pairing

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \times \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) \to \mathbb{C}_K(-1)$$

under which $t_G(\mathbb{C}_K)$ and $t_{G^\vee}(\mathbb{C}_K)$ are orthogonal complements, we get a short exact sequence

$$0 \longrightarrow t_{G^\vee}(\mathbb{C}_K) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \longrightarrow t_G^*(\mathbb{C}_K)(-1) \longrightarrow 0 \qquad (3.8)$$

where all maps are $\mathbb{C}_K$-linear and Γ_K-equivariant. Let us write $d := \dim_{\mathbb{C}_K}(t_G(\mathbb{C}_K))$ and $d^\vee := \dim_{\mathbb{C}_K}(t_{G^\vee}(\mathbb{C}_K))$. We have isomorphisms

$$\mathrm{Ext}^1_{\mathbb{C}_K[\Gamma_K]}(t_G^*(\mathbb{C}_K)(-1), t_{G^\vee}(\mathbb{C}_K)) \simeq \mathrm{Ext}^1_{\mathbb{C}_K[\Gamma_K]}(\mathbb{C}_K(-1)^{\oplus d^\vee}, \mathbb{C}_K^{\oplus d}) \simeq H^1(\Gamma_K, \mathbb{C}_K(1))^{\oplus dd^\vee},$$
$$\mathrm{Hom}_{\mathbb{C}_K[\Gamma_K]}(t_G^*(\mathbb{C}_K)(-1), t_{G^\vee}(\mathbb{C}_K)) \simeq \mathrm{Hom}_{\mathbb{C}_K[\Gamma_K]}(\mathbb{C}_K(-1)^{\oplus d^\vee}, \mathbb{C}_K^{\oplus d}) \simeq H^0(\Gamma_K, \mathbb{C}_K(1))^{\oplus dd^\vee}.$$

Theorem 3.1.14 shows that both $H^0(\Gamma_K, \mathbb{C}_K(1))$ and $H^1(\Gamma_K, \mathbb{C}_K(1))$ vanish. Hence we deduce that the exact sequence (3.8) canonically splits, thereby establishing the desired assertion. □

Definition 3.4.14. Given a p-divisible group G over $\mathcal{O}_K$, we refer to the isomorphism in Theorem 3.4.13 as the *Hodge-Tate decomposition* for G.

COROLLARY 3.4.15. For every p-divisible group G over $\mathcal{O}_K$, the rational Tate-module

$$V_p(G) := T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is a Hodge-Tate p-adic representation of Γ_K.

PROOF. Let us identify the $\mathbb{C}_K$-duals of $t_{G^\vee}(\mathbb{C}_K)$ and $t_G^*(\mathbb{C}_K)$ respectively with $t_{G^\vee}^*(\mathbb{C}_K)$ and $t_G(\mathbb{C}_K)$. Theorem 3.4.13 yields a natural decomposition

$$V_p(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong t_{G^\vee}^*(\mathbb{C}_K) \oplus t_G(\mathbb{C}_K)(1).$$

Therefore we apply Theorem 3.1.14 to find

$$\left(V_p(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \cong \begin{cases} t_{G^\vee}^*(K) & \text{for } n = 0, \\ t_G(K) & \text{for } n = 1, \\ 0 & \text{for } n \neq 0,\ 1. \end{cases}$$

The desired assertion is now evident. □

Remark. Our proof of Corollary 3.4.15 shows that we can find $t_G(K)$ from $T_p(G)$.

PROPOSITION 3.4.16. Let A be an abelian variety over K.

(1) There exists a canonical isomorphism
$$H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \cong \text{Hom}_{\mathbb{Z}_p}(T_p(A[p^\infty]), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$
(2) If A has good reduction, its integral model $\mathcal{A}$ over $\mathcal{O}_K$ yields natural isomorphisms
$$H^0(A, \Omega^1_{A/K}) \cong t^*_{\mathcal{A}[p^\infty]}(K) \quad \text{and} \quad H^1(A, \mathcal{O}_A) \cong t_{\mathcal{A}^\vee[p^\infty]}(K).$$
(3) Given integers $i,\ j \geq 0$ and $n \geq 0$, we have natural identifications
$$H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \cong \bigwedge^n H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p),$$
$$H^i(A, \Omega^j_{A/K}) \cong \bigwedge^i H^1(A, \mathcal{O}_A) \otimes \bigwedge^j H^0(A, \Omega^1_{A/K}).$$

PROOF. All assertions are standard facts about abelian varieties stated in the notes of Milne [**Mil**, §7, §12] and the book of Mumford [**Mum70**, §4]. □

THEOREM 3.4.17. Given an abelian variety A over K with good reduction, there exists have a canonical Γ_K-equivariant isomorphism
$$H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \bigoplus_{i+j=n} H^i(A, \Omega^j_{A/K}) \otimes_K \mathbb{C}_K(-j) \quad \text{for each } n \geq 1.$$

PROOF. Since A has good reduction, it admits an integeral model $\mathcal{A}$ over $\mathcal{O}_K$. We have $T_p(\mathcal{A}[p^\infty]) = T_p(A[p^\infty])$ by definition and find $\mathcal{A}^\vee[p^\infty] \cong \mathcal{A}[p^\infty]^\vee$ by Example 2.1.10. Hence Theorem 3.4.13 and Proposition 3.4.16 together yield a canonical Γ_K-equivariant isomorphism
$$H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong (H^1(A, \mathcal{O}_A) \otimes_K \mathbb{C}_K) \oplus (H^0(A, \Omega^1_{A/K}) \otimes_K \mathbb{C}_K(-1)).$$
Now we deduce the desired assertion from Proposition 3.4.16. □

Remark. Theorem 3.4.17 is a special case of the Hodge-Tate decomposition theorem that we have introduced in Chapter I, Theorem 1.2.2. The proof of the Hodge-Tate decomposition theorem for the general case requires ideas that are beyond the scope of our discussion. We refer curious readers to the notes of Bhatt [**Bha**] for a wonderful exposition of the proof by Scholze [**Sch13**] using perfectoid spaces.

COROLLARY 3.4.18. For every abelian variety A over K with good reduction, the étale cohomology $H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ for each $n \geq 1$ is a Hodge-Tate p-adic representation of Γ_K.

PROOF. Let us take an arbitary integer m. If we have $0 \leq m \leq n$, Theorem 3.1.14 and Theorem 3.4.17 together yield a natural isomorphism
$$\left(H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(m)\right)^{\Gamma_K} \cong H^{n-m}(A, \Omega^m_{A/K}).$$
Otherwise, Theorem 3.1.14 and Theorem 3.4.17 imply that $\left(H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(m)\right)^{\Gamma_K}$ is trivial. Now the desired assertion is straightforward to verify. □

Remark. In fact, given a proper smooth variety X over K, the Hodge-Tate decomposition theorem implies that the étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ for each integer $n \geq 1$ is a Hodge-Tate p-adic representation of Γ_K.

Exercises

1. In this exercise, we study homomorphisms between the R-groups $\mathbb{G}_a$ and $\mathbb{G}_m$.
   (1) Show that every homomorphism from $\mathbb{G}_m$ to $\mathbb{G}_a$ is trivial.
   (2) If R is reduced, show that every homomorphism from $\mathbb{G}_a$ to $\mathbb{G}_m$ is trivial.
   (3) If R contains a nonzero element α with $\alpha^2 = 0$, construct a nonzero homomorphism from $\mathbb{G}_a$ to $\mathbb{G}_m$.

2. Assume that $R = k$ is a field of characteristic p.
   (1) Show that the k-algebra homomorphism $k[t] \to k[t]$ which sends t to $t^p - t$ induces a k-group homomorphism $f : \mathbb{G}_a \to \mathbb{G}_a$.
   (2) Show that $\ker(f)$ is isomorphic to $\underline{\mathbb{Z}/p\mathbb{Z}}$.

3. Prove that an R-group is separated if and only if its unit section is a closed embedding.

Hint. One can identify the unit section as a base change of the diagonal morphism and conversely identify the diagonal morphism as a base change of the unit section.

4. Assume that $R = k$ is a field of characteristic p.
   (1) Verify that the k-group $\alpha_{p^2} := \operatorname{Spec}(k[t]/t^{p^2})$ with the natural additive group structure on $\alpha_{p^2}(B) = \left\{ b \in B : b^{p^2} = 0 \right\}$ for each k-algebra B is finite flat of order p^2.

   (2) Show that $\alpha_{p^2}^\vee$ admits an isomorphism $\alpha_{p^2}^\vee \cong \operatorname{Spec}(k[t,u]/(t^p, u^p))$ with the multiplication on $\alpha_{p^2}^\vee(B) \cong \{ (b_1, b_2) \in B^2 : b_1^p = b_2^p = 0 \}$ for each k-algebra B given by
   $$(b_1, b_2) \cdot (b_1', b_2') = \big(b_1 + b_1', b_2 + b_2' - W_1(b_1, b_2)\big)$$
   where we write $W_1(t,u) := \dfrac{(t+u)^p - t^p - u^p}{p} \in \mathbb{Z}[t,u]$.

   Hint. One can show that a B-algebra homomorphism $B[t, t^{-1}] \to B[t]/(t^{p^2})$ induces a B-group homomorphism $\alpha_{p^2} \to \mathbb{G}_m$ if and only if the image of t is of the form $f(t) = E(b_1 t)E(b_2 t^p)$ with $b_1^p = b_2^p = 0$, where we write $E(t) := \displaystyle\sum_{i=0}^{p-1} \frac{t^i}{i!}$.

   (3) For $k = \overline{\mathbb{F}}_p$, show that α_{p^2} fits into a nonsplit short exact sequence
   $$\underline{0} \longrightarrow \alpha_p \longrightarrow \alpha_{p^2} \longrightarrow \alpha_p \longrightarrow \underline{0}.$$

Remark. For $k = \overline{\mathbb{F}}_p$, there exists a natural identification
$$\operatorname{Ext}^1_{\overline{\mathbb{F}}_p\text{-grp}}(\alpha_p, \alpha_p) \cong (\mathbb{Z}/2\mathbb{Z})^2$$
with elements of $\operatorname{Ext}^1_{\overline{\mathbb{F}}_p\text{-grp}}(\alpha_p, \alpha_p)$ given by α_p^2, α_{p^2}, $\alpha_{p^2}^\vee$, and the p-torsion part of a supersingular elliptic curve. In particular, one can identify the p-torsion part of a supersingular elliptic curve over $\overline{\mathbb{F}}_p$ with the Baer sum of α_p^2 and α_{p^2} as self-extensions of α_p.

5. Assume that $R = k$ is a perfect field.

(1) Given a finite abelian group M with a continuous Γ_k-action, show that the scheme $\underline{M}^{\Gamma_k} := \text{Spec}\,(A)$ for $A := \Big(\prod_{i \in M} \overline{k}\Big)^{\Gamma_k}$ is naturally a finite étale k-group.

Hint. Since M is finite, the Γ_k-action should factor through a finite quotient.

(2) Prove that the inverse functor for the equivalence in Proposition 1.3.4 maps each finite abelian group M with a continuous Γ_k-action to $\underline{M}^{\Gamma_k}$.

(3) Prove that a finite étale group scheme G over a field k is a constant group scheme if and only if the Γ_k-action on $G(\overline{k})$ is trivial.

6. In this exercise, we follow the notes of Pink [**Pin**, §15] to present a counterexample for Proposition 1.4.15 when k is not perfect. Let us choose $c \in k$ which is not a p-th power and set $G := \coprod_{i=0}^{p-1} G_i$ with $G_i := \text{Spec}\,\big(k[t]/(t^p - c^i)\big)$.

(1) Given a k-algebra B, verify a natural identification

$$G_i(B) \cong \big\{\, b \in B : b^p = c^i \,\big\} \qquad \text{for each } i = 0, \cdots, p-1$$

and show that $G(B)$ is a group with multiplication given by the following maps:

- $m_{ij} : G_i(B) \times G_j(B) \to G_{i+j}(B)$ for $i + j < p$ which sends each (g, g') to gg',
- $m_{ij} : G_i(B) \times G_j(B) \to G_{i+j-p}(B)$ for $i + j \geq p$ which sends each (g, g') to gg'/c.

(2) Show that G yields a nonsplit connected-étale sequence

$$\underline{0} \longrightarrow \mu_p \longrightarrow G \longrightarrow \underline{\mathbb{Z}/p\mathbb{Z}} \longrightarrow \underline{0}.$$

Hint. To show that the sequence does not split, compare G_0 with G_i for $i \neq 0$.

7. Assume that $R = k$ is a field.

(1) If k has characteristic 0, establish a natural identification $\text{End}_{k\text{-grp}}(\mathbb{G}_a) \cong k$.

(2) If k has characteristic p, show that $\text{End}_{k\text{-grp}}(\mathbb{G}_a)$ is isomorphic to the (possibly non-commutative) polynomial ring $k\langle \varphi \rangle$ with $\varphi c = c^p \varphi$ for any $c \in k$.

8. Assume that $R = k$ is a field.

(1) Give a proof of Theorem 1.3.10 when $R = k$ is a field without using Theorem 1.1.16.

Hint. If k has characteristic 0, one can adjust the proof of Proposition 1.5.20 to obtain an isomorphism $G^\circ \simeq \text{Spec}\,(k[t_1, \cdots, t_d])$ for some integer $d \geq 0$ and in turn find $d = 0$ by the fact that G° is finite flat.

(2) Prove Theorem 1.1.16 when $R = k$ is a field.

Hint. If k has characteristic 0, one can deduce the assertion from the corresponding theorem for finite groups by observing that G is étale. If k has characteristic p, one can reduce to the case where G is simple with k algebraically closed.

9. Use the self-duality of elliptic curves to prove that every elliptic curve over $\overline{\mathbb{F}}_p$ is either ordinary or supersingular.

10. Assume that $R = k$ is a perfect field.

(1) Show that the dual of every étale p-divisible group over k is connected.

(2) Show that every p-divisible G over k admits a natural decomposition

$$G \cong G^{\mathrm{ll}} \times G^{\mathrm{mult}} \times G^{\text{ét}}$$

with the following properties:

(i) G^{ll} is connected with $(G^{\mathrm{ll}})^\vee$ connected.

(ii) G^{mult} is connected with $(G^{\mathrm{mult}})^\vee$ étale.

(iii) $G^{\text{ét}}$ is étale with $(G^{\text{ét}})^\vee$ connected.

11. Assume that $R = k$ is a field of characteristic 0. Establish an isomorphism between the formal group laws $\mu_{\widehat{\mathbb{G}_a}}$ and $\mu_{\widehat{\mathbb{G}_m}}$ over k defined as in Example 2.2.3.

Hint. Consider the map $k[[t]] \to k[[t]]$ sending t to $\exp(t) - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n!}$.

12. Let K be a finite extension of $\mathbb{Q}_p$ with uniformizer π and residue field $\mathbb{F}_q$.

(1) Show that there exists a unique formal group law μ_π over $\mathcal{O}_K$ of dimension 1 with an endomorphism $[\pi] : \mathcal{O}_K[[t]] \to \mathcal{O}_K[[t]]$ sending t to $\pi t + t^q$.

(2) Show that μ_π is p-divisible.

Remark. The formal group law μ_π is a *Lubin-Tate formal group law*, introduced by the work of Lubin-Tate [**LT65**] as a means to construct the totally ramified abelian extensions of K.

13. For a supersingular elliptic curve E over $\overline{\mathbb{F}}_p$, show that $\ker(\varphi_{E[p]})$ is isomorphic to α_p.

14. Recall that every $\alpha \in \mathbb{Z}_p$ admits a unique p-adic expansion $\alpha = \sum_{n=0}^{\infty} a_n p^n$ where each a_n is an integer with $0 \leq a_n < p$.

(1) Show that the 2-adic expansion agrees with the Teichmüler expansion on $\mathbb{Z}_2$.

(2) Show that the p-adic expansion does not agree with the Teichmüler expansion on $\mathbb{Z}_p$ for $p > 2$.

(3) Find the 3-adic expansion for $[2] \in \mathbb{Z}_3$.

(4) Find the first four coefficients of the 5-adic expansion for $[2] \in \mathbb{Z}_5$.

Hint. The Teichmüler lift of an element $a \in \mathbb{F}_p$ is the unique lift $[a] \in \mathbb{Z}_p$ with $[a]^p = [a]$. One can inductively find its image in $\mathbb{Z}_p/p^n\mathbb{Z}_p = \mathbb{Z}/p^n\mathbb{Z}$ for each $n \geq 1$ by Hensel's lemma.

15. Assume that $R = k$ is a perfect field of characteristic p. For each $\lambda \in \mathbb{Q}$, show that there exists a natural isomorphism $N(\lambda)^\vee \cong N(-\lambda)$.

16. Let A be an abelian variety over $\overline{\mathbb{F}}_p$ of dimension g.

(1) Show that the isocrystal $\mathbb{D}(A[p^\infty])[1/p]$ is self-dual by using the fact that A is isogenous to its dual.

(2) If A is *ordinary* in the sense that $A[p](\overline{\mathbb{F}}_p)$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\oplus g}$, show that there exists an isomorphism

$$A[p^\infty] \simeq (\underline{\mathbb{Q}_p/\mathbb{Z}_p})^g \times (\mu_{p^\infty})^g.$$

Hint. Show that $A[p^\infty]^\circ$ has étale dual, possibly by establishing an isomorphism $\mathbb{D}(A[p^\infty])[1/p] \simeq N(0)^{\oplus g} \oplus N(1)^{\oplus g}$.

17. Let K be a p-adic field.

(1) Prove that its algebraic closure $\overline{K}$ is not p-adically complete.

Hint. There are at least two ways to proceed as follows:

(a) One can observe that $\overline{K}$ is a countable union of nowhere dense subspaces and apply the Baire category theorem to conclude.

(b) Alternatively, one can use Krasner's lemma to produce a Cauchy sequence in $\overline{K}$ whose limit is not algebraic over K.

(2) Prove that $\mathbb{C}_K$ is not discretely valued.

18. Give a proof of Proposition 3.3.10 for $G = \mu_{p^\infty}$.

19. Let K be a p-adic field and E be an elliptic curve over $\mathcal{O}_K$.

(1) Prove that E gives rise to a Γ_K-equivariant $\mathbb{Z}_p$-linear perfect pairing

$$T_p(E[p^\infty]) \times T_p(E[p^\infty]) \to \mathbb{Z}_p(1). \tag{3.9}$$

(2) Deduce that the determinant character of the Γ_K-representation $T_p(E[p^\infty])$ coincides with the p-adic cyclotomic character.

Remark. The perfect pairing (3.9) coincides with the *Weil pairing* on E.

20. Describe the canonical identification

$$\mathrm{Ext}^1_{\mathbb{C}_K[\Gamma_K]}(\mathbb{C}_K(-1), \mathbb{C}_K) \cong H^1(\Gamma_K, \mathbb{C}_K(1))$$

used in the proof of Theorem 3.4.13

Hint. Given a Γ_K-representation V over $\mathbb{C}_K$ with a Γ_K-equivariant short exact sequence

$$0 \longrightarrow \mathbb{C}_K \longrightarrow V \longrightarrow \mathbb{C}_K(-1) \longrightarrow 0,$$

the action of Γ_K on $V(1)$ admits a matrix representation

$$\begin{pmatrix} \chi & c \\ 0 & 1 \end{pmatrix}$$

for some map $c : \Gamma_K \to \mathbb{C}_K(1)$. Show that c is a 1-cocycle on Γ_K in $\mathbb{C}_K(1)$ with its class in $H^1(\Gamma_K, \mathbb{C}_K(1))$ uniquely determined by the isomorphism class of V.

CHAPTER III

Period rings and functors

1. Fontaine's formalism on period rings

The main goal of this section is to discuss the formalism developed by Fontaine [**Fon94**] for p-adic period rings and their associated functors. Our primary references for this section are the notes of Brinon-Conrad [**BC**, §5] and the notes of Fontaine-Oiyang [**FO**, §2.1].

Throughout this chapter, we let K be a p-adic field with absolute Galois group Γ_K, inertia group I_K, and residue field k. In addition, we write $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ for the category of p-adic Γ_K-representations and χ for the p-adic cyclotomic character of K.

1.1. Basic definitions and examples

In this subsection, we define some key notions for our formalism and relate them to Hodge-Tate representations.

Definition 1.1.1. An integral domain B over $\mathbb{Q}_p$ with an action of Γ_K is $(\mathbb{Q}_p, \Gamma_K)$-*regular* if it satisfies the following conditions:

(i) We have $B^{\Gamma_K} = C^{\Gamma_K}$, where C denotes the fraction field of B endowed with a natural Γ_K-action extending the Γ_K-action on B.

(ii) An element $b \in B$ is a unit if $\mathbb{Q}_p \cdot b := \{ c \cdot b : c \in \mathbb{Q}_p \}$ is stable under the Γ_K-action.

Remark. For any field F and any group G, we can similarly define (F, G)-regular rings. The formalism that we develop in this section readily extends to (F, G)-regular rings.

Example 1.1.2. Every field extension of $\mathbb{Q}_p$ with an action of Γ_K is $(\mathbb{Q}_p, \Gamma_K)$-regular.

Definition 1.1.3. Let B be a $(\mathbb{Q}_p, \Gamma_K)$-regular ring with $E := B^{\Gamma_K}$.

(1) We define *the functor associated to* B to be $D_B : \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K) \longrightarrow \mathrm{Vect}_E$ with

$$D_B(V) := (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} \qquad \text{for every } V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K),$$

where Vect_E denotes the category of vector spaces over E.

(2) We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is B-*admissible* if it satisfies the identity

$$\dim_E D_B(V) = \dim_{\mathbb{Q}_p} V.$$

Remark. We can show that the B-admissibility for $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is equivalent to the triviality of the Γ_K-action on $V \otimes_{\mathbb{Q}_p} B$.

Example 1.1.4. We record some examples of admissible representations.

(1) For every $(\mathbb{Q}_p, \Gamma_K)$-regular ring B, trivial Γ_K-representations over $\mathbb{Q}_p$ are B-admissible.

(2) Essentially by Hilbert's Theorem 90, a p-adic representation V of Γ_K is $\overline{K}$-admissible if and only if the action of Γ_K on V factors through a finite quotient.

(3) By a deep result of Sen [**Sen80**], a p-adic representation V of Γ_K is $\mathbb{C}_K$-admissible if and only if the action of I_K on V factors through a finite quotient.

Definition 1.1.5. Given a character $\eta : \Gamma_K \longrightarrow \mathbb{Q}_p^\times$ and a $\mathbb{Q}_p[\Gamma_K]$-module M, we define the *twist of M by η* to be the $\mathbb{Q}_p[\Gamma_K]$-module

$$M(\eta) := M \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta)$$

where $\mathbb{Q}_p(\eta)$ denotes the Γ_K-representation on $\mathbb{Q}_p$ given by η.

Example 1.1.6. Given a $\mathbb{Q}_p[\Gamma_K]$-module M, we have an identification $M(n) \cong M(\chi^n)$ for every $n \in \mathbb{Z}$ by Lemma 3.1.8 in Chapter II.

LEMMA 1.1.7. The group $\chi(I_K)$ is infinite.

PROOF. We have $\ker(\chi) = \bigcap\limits_{v \geq 1} \mathrm{Gal}(K(\mu_{p^v}(\overline{K}))/K)$ as χ encodes the action of Γ_K on $\mathbb{Z}_p(1) = \varprojlim \mu_{p^v}(\overline{K})$. Let us write e_v for the ramification degree of $K(\mu_{p^v}(\overline{K}))$ over K and e for the ramification degree of K over $\mathbb{Q}_p$. We find $e_v e \geq p^{v-1}(p-1)$ by noting that $e_v e$ and $p^{v-1}(p-1)$ are respectively equal to the ramification degrees of $K(\mu_{p^v}(\overline{K}))$ and $\mathbb{Q}_p(\mu_{p^v}(\overline{K}))$ over $\mathbb{Q}_p$. We deduce that e_v grows arbitrarily large and thus obtain the desired assertion. □

THEOREM 1.1.8 (Tate [**Tat67**], Sen [**Sen80**]). Let $\eta : \Gamma_K \longrightarrow \mathbb{Z}_p^\times$ be a continuous character.

(1) If $\eta(I_K)$ is finite, both $H^0(\Gamma_K, \mathbb{C}_K(\eta))$ and $H^1(\Gamma_K, \mathbb{C}_K(\eta))$ are 1-dimensional vector spaces over K.
(2) If $\eta(I_K)$ is infinite, both $H^0(\Gamma_K, \mathbb{C}_K(\eta))$ and $H^1(\Gamma_K, \mathbb{C}_K(\eta))$ vanish.

Remark. Since we have $\mathbb{C}_K(n) \cong \mathbb{C}_K(\chi^n)$ for each $n \in \mathbb{Z}$ as noted in Example 1.1.6, we can deduce Theorem 3.1.14 in Chapter II from Lemma 1.1.7 and Theorem 1.1.8.

Definition 1.1.9. The *Hodge-Tate period ring* is $B_{\mathrm{HT}} := \bigoplus\limits_{n \in \mathbb{Z}} \mathbb{C}_K(n)$.

PROPOSITION 1.1.10. The Hodge-Tate period ring B_{HT} is $(\mathbb{Q}_p, \Gamma_K)$-regular.

PROOF. Let us first show the identity $B_{\mathrm{HT}}^{\Gamma_K} = C_{\mathrm{HT}}^{\Gamma_K}$ for the fraction field C_{HT} of B_{HT}. We consider a natural action of Γ_K on $\mathbb{C}_K((t))$ with $\gamma(t) = \chi(\gamma)t$. Lemma 3.1.8 in Chapter IIyields Γ_K-equivariant isomorphisms $B_{\mathrm{HT}} \simeq \mathbb{C}_K[t, t^{-1}]$ and $C_{\mathrm{HT}} \simeq \mathbb{C}_K(t)$. Since we have $B_{\mathrm{HT}}^{\Gamma_K} = K$ by Theorem 3.1.14 in Chapter II, it suffices to establish the identity $\mathbb{C}_K((t))^{\Gamma_K} = K$. The group Γ_K acts on each $f(t) = \sum c_n t^n \in \mathbb{C}_K((t))$ via the relation

$$\gamma\left(\sum c_n t^n\right) = \sum \gamma(c_n)\chi(\gamma)^n t^n \qquad \text{for every } \gamma \in \Gamma_K.$$

Hence $f(t) = \sum c_n t^n \in \mathbb{C}_K((t))$ is Γ_K-invariant if and only if we have $c_n = \gamma(c_n)\chi(\gamma)^n$ for each $n \in \mathbb{Z}$ and $\gamma \in \Gamma_K$, or equivalently $c_n \in \mathbb{C}_K(n)^{\Gamma_K}$ for every $n \in \mathbb{Z}$ by Lemma 3.1.8 in Chapter II. The desired identity $\mathbb{C}_K((t))^{\Gamma_K} = K$ follows from Theorem 3.1.14 in Chapter II.

It remains to verify that every $b \in B_{\mathrm{HT}}$ with $\mathbb{Q}_p \cdot b$ stable under Γ_K is a unit. Under the isomorphism $B_{\mathrm{HT}} \simeq \mathbb{C}_K[t, t^{-1}]$, we identify b with a function $f(t) = \sum c_n t^n \in \mathbb{C}_K[t, t^{-1}]$. Let us take $m \in \mathbb{Z}$ with $c_m \neq 0$. It suffices to show the identity $c_n = 0$ for each $n \neq m$.

Let $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ be the character that encodes the Γ_K-action on $\mathbb{Q}_p \cdot f(t)$. We note that η is continuous as the Γ_K-action on each $\mathbb{C}_K(n)$ is continuous; in particular, we may regard η as a character with values in $\mathbb{Z}_p^\times$. For each $n \in \mathbb{Z}$ and $\gamma \in \Gamma_K$, we find $\eta(\gamma)c_n = \gamma(c_n)\chi(\gamma)^n$ or equivalently $c_n = (\eta^{-1}\chi^n)(\gamma)\gamma(d_n)$. Hence we have $c_n \in \mathbb{C}_K(\eta^{-1}\chi^n)^{\Gamma_K}$ for every $n \in \mathbb{Z}$ and in turn deduce from Theorem 1.1.8 that $(\eta^{-1}\chi^n)(I_K)$ is finite for every $n \in \mathbb{Z}$ with $c_n \neq 0$.

Suppose for contradiction that we have $c_n \neq 0$ for some $n \neq m$. Since both $(\eta^{-1}\chi^n)(I_K)$ and $(\eta^{-1}\chi^m)(I_K)$ are finite, I_K has a finite image under $\chi^{n-m} = (\eta^{-1}\chi^n) \cdot (\eta^{-1}\chi^m)^{-1}$. Hence we obtain a desired contradiction by Lemma 1.1.7, thereby completing the proof. □

PROPOSITION 1.1.11. A p-adic representation V of Γ_K is Hodge-Tate if and only if it is B_{HT}-admissible.

PROOF. Since we have

$$D_{B_{\mathrm{HT}}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{\Gamma_K} = \bigoplus_{n \in \mathbb{Z}} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K}, \tag{1.1}$$

the desired assertion follows from Lemma 3.1.15 in Chapter II. □

Example 1.1.12. Given a p-adic Γ_K-representation V which fits into an exact sequence

$$0 \longrightarrow \mathbb{Q}_p(m) \longrightarrow V \longrightarrow \mathbb{Q}_p(n) \longrightarrow 0$$

with $m \neq n$, we assert that V is Hodge-Tate. For every $i \in \mathbb{Z}$, we have an exact sequence

$$0 \longrightarrow \mathbb{C}_K(i+m) \longrightarrow V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(i) \longrightarrow \mathbb{C}_K(i+n) \longrightarrow 0$$

by the flatness of $\mathbb{C}_K(i)$ over $\mathbb{Q}_p$ and consequently obtain a long exact sequence

$$0 \longrightarrow \mathbb{C}_K(i+m)^{\Gamma_K} \longrightarrow (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(i))^{\Gamma_K} \longrightarrow \mathbb{C}_K(i+n)^{\Gamma_K} \longrightarrow H^1(\Gamma_K, \mathbb{C}_K(i+m)).$$

Therefore Theorem 3.1.14 in Chapter II yields an identification

$$(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(i))^{\Gamma_K} \cong \begin{cases} K & \text{for } i = -m, -n, \\ 0 & \text{for } i \neq -m, -n. \end{cases}$$

Now we find

$$\dim_K D_{B_{\mathrm{HT}}}(V) = \sum_{i \in \mathbb{Z}} \dim_K (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(i))^{\Gamma_K} = 2 = \dim_{\mathbb{Q}_p} V$$

and in turn establish the desired assertion.

Remark. On the other hand, a self extension of $\mathbb{Q}_p$ is not necessarily Hodge-Tate. For example, by a difficult result of Sen [**Sen80**], the two-dimensional vector space over $\mathbb{Q}_p$ where each $\gamma \in \Gamma_K$ acts via the matrix $\begin{pmatrix} 1 & \log_p(\chi(\gamma)) \\ 0 & 1 \end{pmatrix}$ is not Hodge-Tate.

PROPOSITION 1.1.13. Given a continuous character $\eta : \Gamma_K \longrightarrow \mathbb{Z}_p^\times$, the Γ_K-representation $\mathbb{Q}_p(\eta)$ is Hodge-Tate if and only if there exists some $n \in \mathbb{Z}$ with $(\eta\chi^n)(I_K)$ finite.

PROOF. By Lemma 3.1.15 in Chapter II, the 1-dimensional Γ_K-representation $\mathbb{Q}_p(\eta)$ is Hodge-Tate if and only if there exists some $n \in \mathbb{Z}$ with $(\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} \neq 0$, or equivalently $\mathbb{C}_K(\eta\chi^n)^{\Gamma_K} \neq 0$ by Example 1.1.6. Hence the assertion follows from Theorem 1.1.8. □

Definition 1.1.14. Given a Hodge-Tate representation V, an integer $n \in \mathbb{Z}$ is a *Hodge-Tate weight* of V with multiplicity m if we have

$$\dim_K (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} = m > 0.$$

Remark. Readers should be aware that many authors use the opposite sign convention for Hodge-Tate weights. We will explain the reason for our choice in §2.4.

Example 1.1.15. We record the Hodge-Tate weights for some Hodge-Tate representations.

(1) For every $n \in \mathbb{Z}$, the Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ has Hodge-Tate weight $-n$.
(2) For every p-divisible group G over $\mathcal{O}_K$, the rational Tate module $V_p(G)$ has Hodge-Tate weights 0 or -1 (possibly both) by Theorem 3.4.13 in Chapter II.
(3) For an abelian variety A over K with good reduction, the étale cohomology $H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ has Hodge-Tate weights $0, 1, \cdots, n$ as easily seen by Theorem 3.4.17 in Chapter II.

1.2. Formal properties of admissible representations

Throughout this subsection, we fix a $(\mathbb{Q}_p, \Gamma_K)$-regular ring B and write $E := B^{\Gamma_K}$. In addition, we denote by $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_K)$ the category of B-admissible p-adic Γ_K-representations.

THEOREM 1.2.1. Let V be a p-adic Γ_K-representation.

(1) There exists a natural map
$$\alpha_V : D_B(V) \otimes_E B \longrightarrow V \otimes_{\mathbb{Q}_p} B$$
which is B-linear, Γ_K-equivariant, and injective.

(2) V satisfies the inequality
$$\dim_E D_B(V) \leq \dim_{\mathbb{Q}_p} V \tag{1.2}$$
with equality precisely when α_V is an isomorphism.

PROOF. Let us first consider statement (1). We have the natural map
$$\alpha_V : D_B(V) \otimes_E B \longrightarrow (V \otimes_{\mathbb{Q}_p} B) \otimes_E B \cong V \otimes_{\mathbb{Q}_p} (B \otimes_E B) \longrightarrow V \otimes_{\mathbb{Q}_p} B,$$
which is clearly B-linear and Γ_K-equivariant. We wish to show that α_V is injective. Since the fraction field C of B is $(\mathbb{Q}_p, \Gamma_K)$-regular as noted in Example 1.1.2, we obtain a natural map
$$\beta_V : D_C(V) \otimes_E C \longrightarrow V \otimes_{\mathbb{Q}_p} C$$
which fits into a commutative diagram
$$\begin{array}{ccc} D_B(V) \otimes_E B & \xrightarrow{\alpha_V} & V \otimes_{\mathbb{Q}_p} B \\ \big\downarrow & & \big\downarrow \\ D_C(V) \otimes_E C & \xrightarrow{\beta_V} & V \otimes_{\mathbb{Q}_p} C \end{array}$$
with injective vertical maps. It suffices to prove that β_V is injective. Suppose for contradiction that $\ker(\beta_V)$ is not trivial. Let us take an E-basis (e_i) of $D_C(V) = (V \otimes_{\mathbb{Q}_p} C)^{\Gamma_K}$ and regard each e_i as a vector in $V \otimes_{\mathbb{Q}_p} C$. By our assumption, there exists a nontrivial C-linear relation $\sum c_i e_i = 0$ with minimal number of nonzero terms. Without loss of generality, we may set $c_j = 1$ for some j. For every $\gamma \in \Gamma_K$ we find
$$0 = \gamma\left(\sum c_i e_i\right) - \sum c_i e_i = \sum (\gamma(c_i) - c_i) e_i.$$
Since the coefficient of e_j is zero, the minimality of our relation yields the identity $c_i = \gamma(c_i)$ for each c_i and in turn implies that c_i lies in $C^{\Gamma_K} = E$. Hence we have a nontrivial E-linear relation $\sum c_i e_i = 0$ for the E-basis (e_i) of $D_C(V)$, thereby obtaining a desired contradiction.

It remains to verify statement (2). Since the inequality (1.2) is evident by statement (1), we only need to consider the equality condition. If α_V is an isomorphism, the inequality becomes an equality. For the converse, we henceforth assume the identity $\dim_E D_B(V) = \dim_{\mathbb{Q}_p} V$. Let us choose a basis (u_i) of $D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ over E and a basis (v_i) of V over $\mathbb{Q}_p$. We may represent α_V by a $d \times d$ matrix M_V with $d := \dim_E D_B(V) = \dim_{\mathbb{Q}_p} V$. We wish to show that $\det(M_V)$ is a unit in B. We have $\det(M_V) \neq 0$ as the map $D_B(V) \otimes_E C \to V \otimes_{\mathbb{Q}_p} C$ induced by α_V is an isomorphism for being an injective map between vector spaces of equal dimension. Let us consider the identity $(\wedge^d \alpha_V)(u_1 \wedge \cdots \wedge u_d) = \det(M_V)(v_1 \wedge \cdots \wedge v_d)$. The group Γ_K acts trivially on $u_1 \wedge \cdots \wedge u_d$ and by some $\mathbb{Q}_p$-valued character η on $v_1 \wedge \cdots \wedge v_d$. Since α_V is Γ_K-equivariant, we deduce that Γ_K acts on $\det(M_V)$ by η^{-1}. Hence we find $\det(M_V) \in B^\times$ as B is $(\mathbb{Q}_p, \Gamma_K)$-regular, thereby completing the proof. $\square$

Proposition 1.2.2. The functor D_B is exact and faithful on $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$.

Proof. Let V and W be arbitrary B-admissible representations. Theorem 1.2.1 yields natural Γ_K-equivariant B-linear isomorphisms

$$D_B(V) \otimes_E B \cong V \otimes_{\mathbb{Q}_p} B \quad \text{and} \quad D_B(W) \otimes_E B \cong W \otimes_{\mathbb{Q}_p} B.$$

Given $f \in \mathrm{Hom}_{\mathbb{Q}_p[\Gamma_K]}(V, W)$ with the associated map $D_B(f) : D_B(V) \to D_B(W)$ being zero, we observe that the map $V \otimes_{\mathbb{Q}_p} B \to W \otimes_{\mathbb{Q}_p} B$ induced by f is zero and in turn deduce that f must be zero. Therefore the functor D_B is faithful on $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$.

It remains to verify that D_B is exact on $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$. Let us consider an arbitrary short exact sequence of B-admissible representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$$

Since every algebra over a field is faithfully flat, B is faithfully flat over both $\mathbb{Q}_p$ and E. Therefore we obtain a short exact sequence

$$0 \longrightarrow U \otimes_{\mathbb{Q}_p} B \longrightarrow V \otimes_{\mathbb{Q}_p} B \longrightarrow W \otimes_{\mathbb{Q}_p} B \longrightarrow 0,$$

which we naturally identify with a short exact sequence

$$0 \longrightarrow D_B(U) \otimes_E B \longrightarrow D_B(V) \otimes_E B \longrightarrow D_B(W) \otimes_E B \longrightarrow 0$$

by Theorem 1.2.1. The desired assertion is now evident as B is faithfully flat over E. □

Remark. The functor D_B is not fully faithful on $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ with values in the category of vector spaces over E; indeed, the isomorphism class of $D_B(V)$ for every $V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ depends only on the dimension of V. In practice, however, we enhance D_B to a functor that takes values in a category of vector spaces over E with some additional structures, as briefly described in Chapter I, §1.3. We will see in §3 that such an enhanced functor is fully faithful for the crystaline period ring $B = B_{\mathrm{cris}}$.

Proposition 1.2.3. The category $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ is closed under taking subquotients.

Proof. Consider a short exact sequence of p-adic representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0 \tag{1.3}$$

with $V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$. We wish to show that both U and W are B-admissible. Since the functor D_B is left exact on $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ by construction, we have an exact sequence

$$0 \longrightarrow D_B(U) \longrightarrow D_B(V) \longrightarrow D_B(W). \tag{1.4}$$

In addition, by Theorem 1.2.1 we have

$$\dim_E D_B(U) \leq \dim_{\mathbb{Q}_p} U \quad \text{and} \quad \dim_E D_B(W) \leq \dim_{\mathbb{Q}_p} W. \tag{1.5}$$

Now the exact sequences (1.3) and (1.4) together yield inequalities

$$\dim_E D_B(V) \leq \dim_E D_B(U) + \dim_E D_B(W) \leq \dim_{\mathbb{Q}_p} U + \dim_{\mathbb{Q}_p} W = \dim_{\mathbb{Q}_p} V.$$

Since V is B-admissible, all inequalities are in fact equalities. Therefore we deduce that both U and W are B-admissible as desired. □

Remark. In general, the category $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ is not closed under taking extensions. For example, the category of Hodge-Tate representations is not closed under taking extensions by the remark following Example 1.1.12.

Proposition 1.2.4. Given $V,\ W \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$, we have $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ with a natural isomorphism

$$D_B(V) \otimes_E D_B(W) \cong D_B(V \otimes_{\mathbb{Q}_p} W).$$

Proof. Theorem 1.2.1 yields natural Γ_K-equivariant B-linear isomorphisms

$$\alpha_V : D_B(V) \otimes_E B \xrightarrow{\sim} V \otimes_{\mathbb{Q}_p} B \quad \text{and} \quad \alpha_W : D_B(W) \otimes_E B \xrightarrow{\sim} W \otimes_{\mathbb{Q}_p} B.$$

Let us consider the natural map

$$D_B(V) \otimes_E D_B(W) \longrightarrow (V \otimes_{\mathbb{Q}_p} B) \otimes_E (W \otimes_{\mathbb{Q}_p} B) \longrightarrow (V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B \tag{1.6}$$

with the first arrow given by the identifications

$$D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} \quad \text{and} \quad D_B(W) = (W \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}.$$

Since the second arrow is evidently Γ_K-equivariant, we obtain a natural E-linear map

$$D_B(V) \otimes_E D_B(W) \longrightarrow \big((V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B\big)^{\Gamma_K} \cong D_B(V \otimes_{\mathbb{Q}_p} W). \tag{1.7}$$

This map is injective since the map (1.6) extends to a B-linear map

$$(D_B(V) \otimes_E D_B(W)) \otimes_E B \longrightarrow \big((V \otimes_{\mathbb{Q}_p} B) \otimes_E (W \otimes_{\mathbb{Q}_p} B)\big) \otimes_E B \longrightarrow (V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B$$

which coincides with the isomorphism $\alpha_V \otimes \alpha_W$ under the identifications

$$(D_B(V) \otimes_E D_B(W)) \otimes_E B \cong (D_B(V) \otimes_E B) \otimes_B (D_B(W) \otimes_E B),$$
$$\big((V \otimes_{\mathbb{Q}_p} B) \otimes_E (W \otimes_{\mathbb{Q}_p} B)\big) \otimes_E B \cong (V \otimes_{\mathbb{Q}_p} B \otimes_E B) \otimes_B (W \otimes_{\mathbb{Q}_p} B \otimes_E B),$$
$$(V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B \cong (V \otimes_{\mathbb{Q}_p} B) \otimes_B (W \otimes_{\mathbb{Q}_p} B).$$

Therefore we find

$$\dim_E D_B(V \otimes_{\mathbb{Q}_p} W) \geq (\dim_E D_B(V)) \cdot (\dim_E D_B(W)) = \dim_{\mathbb{Q}_p} V \otimes_{\mathbb{Q}_p} W$$

where the equality follows from the B-admissibility of V and W. We see by Theorem 1.2.1 that this inequality is indeed an equality and in turn deduce that $V \otimes_{\mathbb{Q}_p} W$ is a B-admissible representation with the natural isomorphism (1.7). □

Proposition 1.2.5. For every $V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$, we have $\wedge^n(V),\ \mathrm{Sym}^n V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ with natural isomorphisms

$$\wedge^n(D_B(V)) \cong D_B(\wedge^n(V)) \qquad \text{and} \qquad \mathrm{Sym}^n(D_B(V)) \cong D_B(\mathrm{Sym}^n(V)).$$

Proof. Let us only consider exterior powers here, as the same argument works with symmetric powers. Proposition 1.2.4 implies that $V^{\otimes n}$ is B-admissible with a natural isomorphism $D_B(V^{\otimes n}) \cong D_B(V)^{\otimes n}$. We find $\wedge^n(V) \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ by Proposition 1.2.3 and in turn obtain a natural surjective E-linear map

$$D_B(V)^{\otimes n} \xrightarrow{\sim} D_B(V^{\otimes n}) \twoheadrightarrow D_B(\wedge^n(V))$$

by Proposition 1.2.2. It is straightforward to check that this map factors through the natural surjection $D_B(V)^{\otimes n} \twoheadrightarrow \wedge^n(D_B(V))$. Hence we have a natural surjective E-linear map

$$\wedge^n(D_B(V)) \twoheadrightarrow D_B(\wedge^n(V)),$$

which turns out to be an isomorphism since we have

$$\dim_E \wedge^n(D_B(V)) = \dim_E D_B(\wedge^n(V))$$

by the B-admissibility of V and $\wedge^n(V)$. □

PROPOSITION 1.2.6. For every $V \in \mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$, we have $V^\vee \in \mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$ with the natural E-linear map

$$D_B(V) \otimes_E D_B(V^\vee) \cong D_B(V \otimes_{\mathbb{Q}_p} V^\vee) \longrightarrow D_B(\mathbb{Q}_p) \cong E \tag{1.8}$$

being a perfect pairing.

PROOF. Let us first consider the case where V has dimension 1 over $\mathbb{Q}_p$. We fix a nonzero vector $v \in V$ and take $f \in V^\vee = \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)$ with $f(v) = 1$. We represent the Γ_K-action on V by a continuous character $\eta : \Gamma_K \longrightarrow \mathbb{Q}_p^\times$ and obtain the relations

$$\gamma(v) = \eta(\gamma)v \quad \text{ and } \quad \gamma(f) = \eta(\gamma)^{-1}f \quad \text{ for every } \gamma \in \Gamma_K.$$

Since $D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ is 1-dimensional over E by the B-admissibility of V, it admits a basis given by a vector $v \otimes b$ for some $b \in B$. Now we find

$$v \otimes b = \gamma(v \otimes b) = \gamma(v) \otimes \gamma(b) = \eta(\gamma)v \otimes \gamma(b) = v \otimes \eta(\gamma)\gamma(b) \quad \text{ for every } \gamma \in \Gamma_K$$

or equivalently

$$b = \eta(\gamma)\gamma(b) \qquad \text{ for every } \gamma \in \Gamma_K.$$

Moreover, we have $b \in B^\times$ as $v \otimes b$ yields a B-basis for $V \otimes_{\mathbb{Q}_p} B$ via the natural isomorphism $D_B(V) \otimes_E B \cong V \otimes_{\mathbb{Q}_p} B$ given by Theorem 1.2.1. Hence $D_B(V^\vee) = (V^\vee \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ contains a nonzero vector $f \otimes b^{-1}$. We deduce that the inequality

$$\dim_E D_B(V^\vee) \leq \dim_{\mathbb{Q}_p} V^\vee = 1$$

given by Theorem 1.2.1 must be an equality, which means that $V^\vee$ is B-admissible. We also observe that $f \otimes b^{-1}$ yields an E-basis for $D_B(V^\vee)$ and in turn find that the map (1.8) is a perfect pairing.

We now establish the B-admissibility of $V^\vee$ in the general case. Let us write $d := \dim_{\mathbb{Q}_p} V$ for notational convenience. We have a natural Γ_K-equivariant isomorphism

$$\Delta : \det(V^\vee) \otimes_{\mathbb{Q}_p} \wedge^{d-1} V \xrightarrow{\sim} V^\vee$$

given by the relation

$$\Delta\left((f_1 \wedge \cdots \wedge f_d) \otimes (v_2 \wedge \cdots \wedge v_d)\right)(v_1) = \det(f_i(v_j)) \quad \text{ for all } f_i \in V^\vee \text{ and } v_j \in V.$$

Proposition 1.2.5 implies that both $\det(V) = \wedge^d V$ and $\wedge^{d-1} V$ are B-admissible. Moreover, our discussion in the preceding paragraph shows that $\det(V^\vee) \cong \det(V)^\vee$ is also B-admissible as $\det(V)$ has dimension 1 over $\mathbb{Q}_p$. Therefore $V^\vee$ is B-admissible by Proposition 1.2.4.

It remains to prove that the map (1.8) is a perfect pairing in the general case. Since both V and $V^\vee$ are B-admissible, we find

$$d = \dim_E D_B(V) = \dim_E D_B(V^\vee).$$

Upon choosing E-bases for $D_B(V)$ and $D_B(V^\vee)$, we can represent the pairing (1.8) by a $d \times d$ matrix M. It suffices to show that $\det(M)$ is not zero or equivalently that the induced pairing

$$\det(D_B(V)) \otimes_E \det(D_B(V^\vee)) \longrightarrow E$$

is perfect. Since we have natural identifications

$$\det(D_B(V)) \cong D_B(\det(V)) \quad \text{ and } \quad \det(D_B(V^\vee)) \cong D_B(\det(V^\vee))$$

given by Proposition 1.2.5, the desired assertion is evident by our discussion in the first paragraph. $\square$

2. de Rham representations

In this section, we define and study the de Rham period ring and de Rham representations. The primary references for this section are the notes of Brinon-Conrad [**BC**, §4 and §6] and the article of Scholze [**Sch12**].

2.1. Perfectoid fields and their tilts

Let us begin with the notion of perfectoid fields, which provides a modern perspective of Fontaine's original work.

Definition 2.1.1. A *perfectoid field* is a complete nonarchimedean field C of residue characteristic p with the following properties:

(i) The valuation on C is nondiscrete.
(ii) The p-th power map on $\mathcal{O}_C/p\mathcal{O}_C$ is surjective.

Remark. By convention, we assume that the valuation on a nonarchimedean field is not trivial. On the other hand, the valuation on a valued field may be trivial.

LEMMA 2.1.2. Let C be a complete nonarchimedean field of residue characteristic p. If the p-th power map on C is surjective, the field C is a perfectoid field.

PROOF. Let us denote by ν the valuation on C and take an arbitrary element $x \in C$. Since the p-th power map on C is surjective by our assumption, there exists an element $y \in C$ with $x = y^p$. If x has positive valuation, we find

$$0 < \nu(y) = \nu(x)/p < \nu(x). \tag{2.1}$$

We deduce that C does not have an element with minimum positive valuation, which in particular implies that the valuation ν is not discrete. In addition, we observe that the p-th power map on $\mathcal{O}_C$ is surjective; indeed, if x lies in $\mathcal{O}_C$ we have $x = y^p$ with $y \in \mathcal{O}_C$ by the relation (2.1). Hence the p-th power map on $\mathcal{O}_C/p\mathcal{O}_C$ is also surjective. The desired assertion is now evident. □

Remark. The converse of Lemma 2.1.2 does not hold; in other words, the p-th power map on a perfectoid field is not neccessarily surjective.

Example 2.1.3. Since $\mathbb{C}_K$ is algebraically closed as noted in Chapter II, Proposition 3.1.13, it is a perfectoid field by Lemma 2.1.2.

Remark. In fact, Lemma 2.1.2 shows that every complete nonarchimedean algebraically closed field of residue characteristic p is a perfectoid field.

PROPOSITION 2.1.4. A nonarchimedean field of characteristic p is perfectoid if and only if it is complete and perfect.

PROOF. By definition, every perfectoid field of characteristic p is complete and perfect. Conversely, every complete nonarchimedean perfect field of characteristic p is perfectoid by Lemma 2.1.2. □

Definition 2.1.5. Let C be a perfectoid field.

(1) The *tilt* of C is $C^\flat := \varprojlim_{x \mapsto x^p} C$ endowed with the natural multiplication.
(2) The *sharp map* associated to C is the map $C^\flat \to C$ which sends each $c = (c_n) \in C^\flat$ to the first component $c^\sharp := c_0$.

For the rest of this subsection, we fix a perfectoid field C with the valuation ν. Its tilt $C^\flat$ is *a priori* a multiplicative monoid. We aim to show that $C^\flat$ is naturally a perfectoid field of characteristic p.

PROPOSITION 2.1.6. Fix an element $\varpi \in C^\times$ with $0 < \nu(\varpi) \leq \nu(p)$.

(1) Given arbitrary elements $x, y \in \mathcal{O}_C$ with $x - y \in \varpi\mathcal{O}_C$ we have
$$x^{p^n} - y^{p^n} \in \varpi^{n+1}\mathcal{O}_C \qquad \text{for each integer } n \geq 0.$$
(2) The natural projection $\mathcal{O}_C \twoheadrightarrow \mathcal{O}_C/\varpi\mathcal{O}_C$ induces a multiplicative bijection
$$\varprojlim_{x \mapsto x^p} \mathcal{O}_C \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C. \tag{2.2}$$
(3) The monoid $\varprojlim_{x \mapsto x^p} \mathcal{O}_C$ is naturally a ring of characteristic p via the map (2.2).

PROOF. The inequality $\nu(\varpi) \leq \nu(p)$ implies that p is divisible by ϖ in $\mathcal{O}_C$. In addition, for elements $x, y \in \mathcal{O}_C$ and an integer $n \geq 1$ we find
$$x^{p^n} - y^{p^n} = \left(y^{p^{n-1}} + (x^{p^{n-1}} - y^{p^{n-1}})\right)^p - y^{p^n} \qquad \text{for each } n \geq 1.$$
Hence we obtain statement (1) by a simple induction.

Let us now consider statement (2). We wish to construct an inverse map
$$f : \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C \longrightarrow \varprojlim_{x \mapsto x^p} \mathcal{O}_C.$$
Take an arbitrary element $\overline{c} = (\overline{c}_n) \in \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$ and choose a lift $c_n \in \mathcal{O}_C$ of each $\overline{c}_n$. We have
$$c_{n+m+l}^{p^l} - c_{n+m} \in \varpi\mathcal{O}_C \qquad \text{for all } l, m, n \geq 0,$$
and consequently find
$$c_{n+m+l}^{p^{m+l}} - c_{n+m}^{p^m} \in \varpi^{m+1}\mathcal{O}_C \qquad \text{for all } n, m \geq 0$$
by statement (1). Hence for each $n \geq 0$ the sequence $(c_{n+m}^{p^m})_{m \geq 0}$ converges in $\mathcal{O}_C$ for being Cauchy. In addition, statement (1) implies that the limit of the sequence $(c_{n+m}^{p^m})_{m \geq 0}$ for each $n \geq 0$ does not depend on the choice of c_n. Now we write
$$f_n(\overline{c}) := \lim_{m \to \infty} c_{n+m}^{p^m} \qquad \text{for each } n \geq 0$$
and obtain the desired inverse by setting
$$f(\overline{c}) := (f_n(\overline{c})) \in \varprojlim_{x \mapsto x^p} \mathcal{O}_C.$$

It remains to verify statement (3). Since ϖ divides p in $\mathcal{O}_C$ as already noted in the first paragraph, the ring $\mathcal{O}_C/\varpi\mathcal{O}_C$ is of characteristic p and thus induces a natural ring structure on $\varprojlim_{x \mapsto x^p} \mathcal{O}_C \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$. Moreover, this ring structure does not depend on ϖ; indeed, for arbitrary elements $a = (a_n)$ and $b = (b_n)$ in $\varprojlim_{x \mapsto x^p} \mathcal{O}_C$ we find
$$ab = (a_n b_n) \qquad \text{and} \qquad a + b = \left(\lim_{m \to \infty} (a_{m+n} + b_{m+n})^{p^m}\right).$$
Now we establish statement (3) as $\varprojlim_{x \mapsto x^p} \mathcal{O}_C$ is evidently of characteristic p. $\square$

Proposition 2.1.7. The tilt $C^\flat$ of C is naturally a field of characteristic p which is complete with respect to the valuation $\nu^\flat$ given by $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in C^\flat$ with $\mathcal{O}_{C^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_C$.

Proof. Let us fix an element $\varpi \in C^\times$ with $0 < \nu(\varpi) \leq \nu(p)$. Proposition 2.1.6 shows that $\mathcal{O} := \varprojlim_{x \mapsto x^p} \mathcal{O}_C$ is naturally a ring of characteristic p with a canonical identification

$$\mathcal{O} \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C. \tag{2.3}$$

We may identify $C^\flat$ with the fraction field of $\mathcal{O}$, which is evidently perfect of characteristic p.

We assert that the function $\nu^\flat$ on $C^\flat$ with $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in C^\flat$ is indeed a valuation. It is clear by construction that $\nu^\flat$ is a multiplicative homomorphism. Let us take arbitrary elements $a = (a_n)$ and $b = (b_n)$ in $C^\flat$. Without loss of generality, we may assume $\nu^\flat(a) \geq \nu^\flat(b)$ or equivalently $\nu(a_0) \geq \nu(b_0)$. Since we have

$$\nu(a_n) = \frac{1}{p^n}\nu(a_0) \geq \frac{1}{p^n}\nu(b_0) = \nu(b_n) \quad \text{ for each } n \geq 0,$$

we may write $a = bu$ for some $u \in \mathcal{O}$ and find

$$\nu^\flat(a+b) = \nu^\flat((u+1)b) = \nu^\flat(u+1) + \nu^\flat(b) \geq \nu^\flat(b) = \min(\nu^\flat(a), \nu^\flat(b))$$

where the inequality follows from the observation that $u+1$ is an element of $\mathcal{O}$. Therefore we deduce that $\nu^\flat$ is a valuation.

Let us now take an arbitrary element $c = (c_n) \in C^\flat$. We find

$$\nu(c_n) = \frac{1}{p^n}\nu(c_0) = \frac{1}{p^n}\nu^\flat(c) \quad \text{ for each } n \geq 0$$

and in turn verify that $\mathcal{O}$ is indeed the valuation ring of $C^\flat$. Moreover, given an arbitrary integer $m > 0$ we have $\nu(c_n) \geq \nu(\varpi)$ for each $n \leq m$ if and only if c satisfies the inequality $\nu^\flat(c) \geq p^m\nu(\varpi)$. Hence the isomorphism (2.3) is a homeomorphism with $\mathcal{O}$ and $\varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$ respectively endowed with the $\nu^\flat$-adic topology and the inverse limit topology. It is not hard to see that $\varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$ is complete, which consequently implies that both $\mathcal{O}_{C^\flat} = \mathcal{O}$ and $C^\flat$ are complete. □

Remark. Proposition 2.1.6 and Proposition 2.1.7 remain valid if we replace C by an arbitrary complete nonarchimedean field L with its "tilt" $L^\flat := \varprojlim_{c \mapsto c^p} L$. However, if L is not perfectoid the valuation on $L^\flat$ may be trivial. For example, if L is a p-adic field $L^\flat$ is isomorphic to its residue field with the trivial valuation.

Lemma 2.1.8. For every $c \in \mathcal{O}_C$ there exists an element $c^\flat \in \mathcal{O}_{C^\flat}$ with $c - (c^\flat)^\sharp \in p\mathcal{O}_C$.

Proof. Proposition 2.1.6 and Proposition 2.1.7 together yield a natural isormohpsim

$$\mathcal{O}_{C^\flat} \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p\mathcal{O}_C$$

Let $\overline{c}$ denote the image of c in $\mathcal{O}_C/p\mathcal{O}_C$. Since the p-th power map on $\mathcal{O}_C/p\mathcal{O}_C$ is surjective, we obtain the desired assertion by taking $c^\flat = (c_n^\flat) \in \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p\mathcal{O}_C \cong \mathcal{O}_{C^\flat}$ with $c_0^\flat = \overline{c}$. □

Remark. Such an element $c^\flat \in \mathcal{O}_{C^\flat}$ is not unique unless C is of characteristic p.

PROPOSITION 2.1.9. The map $\mathcal{O}_{C^\flat} \to \mathcal{O}_C/p\mathcal{O}_C$ which sends each $c \in \mathcal{O}_{C^\flat}$ to the image of $c^\sharp$ in $\mathcal{O}_C/p\mathcal{O}_C$ is a surjective ring homomorphism.

PROOF. Since we have $\mathcal{O}_{C^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_C$ as noted in Proposition 2.1.7, the assertion is straightforward to verify by Proposition 2.1.6 and Lemma 2.1.8. □

Remark. The sharp map associated to C is a multiplicative map but is not a ring homomorphism unless C is of characteristic p.

PROPOSITION 2.1.10. The valued fields C and $C^\flat$ have the same value groups.

PROOF. Let $\nu^\flat$ denote the valuation on $C^\flat$. Since we have $\nu^\flat\left((C^\flat)^\times\right) \subseteq \nu(C^\times)$ by Proposition 2.1.7, we only need to show the relation $\nu(C^\times) \subseteq \nu^\flat\left((C^\flat)^\times\right)$. Let us consider an arbitrary element $c \in C^\times$. We wish to find an element $c^\flat \in (C^\flat)^\times$ with $\nu^\flat(c^\flat) = \nu(c)$. Since ν is nondiscrete, we can choose an element $\varpi \in \mathcal{O}_C$ with $0 < \nu(\varpi) < \nu(p)$. Let us write $c = \varpi^n u$ for some $n \in \mathbb{Z}$ and $u \in \mathcal{O}_C$ with $\nu(u) < \nu(\varpi)$. Lemma 2.1.8 yields elements $\varpi^\flat$ and $u^\flat$ in $\mathcal{O}_{C^\flat}$ with $\varpi - (\varpi^\flat)^\sharp \in p\mathcal{O}_C$ and $u - (u^\flat)^\sharp \in p\mathcal{O}_C$. By Proposition 2.1.7 we find

$$\nu^\flat(\varpi^\flat) = \nu((\varpi^\flat)^\sharp) = \nu\left(\varpi - (\varpi - (\varpi^\flat)^\sharp)\right) = \nu(\varpi),$$
$$\nu^\flat(u^\flat) = \nu((u^\flat)^\sharp) = \nu\left(u - (u - (u^\flat)^\sharp)\right) = \nu(u).$$

Hence we obtain the desired assertion by taking $c^\flat = (\varpi^\flat)^n u^\flat$. □

PROPOSITION 2.1.11. The field $C^\flat$ is a perfectoid field of characteristic p.

PROOF. Proposition 2.1.10 implies that the value group of $C^\flat$ is not trivial. Since $C^\flat$ is perfect by construction, the assertion follows from Proposition 2.1.4 and Proposition 2.1.7. □

Remark. Scholze [**Sch12**] shows that C and $C^\flat$ satisfy the following additional properties:

(i) Every finite extension of C is perfectoid.
(ii) There exists a canonical bijection

$$\{ \text{ Finite extensions of } C \,\} \xrightarrow{\sim} \left\{ \text{ Finite extensions of } C^\flat \right\}$$

which sends each finite extension L of C to its tilt $L^\flat$.
(iii) The residue fields of C and $C^\flat$ are naturally isomorphic.

Example 2.1.12. Since $\mathbb{C}_K$ is a perfectoid field as noted in Example 2.1.3, its tilt $F := \mathbb{C}_K^\flat$ is a perfectoid field of characteristic p by Proposition 2.1.11.

Remark. Since $\mathbb{C}_K$ is algebraically closed as noted in Chapter II, Proposition 3.1.13, the remark after Proposition 2.1.11 shows that F is algebraically closed. We will present a proof of this fact in Chapter IV. If K is a finite extension of $\mathbb{Q}_p$, we can naturally identify $F = \mathbb{C}_K^\flat$ with the t-adic completion of $\overline{\mathbb{F}_p((t))}$.

PROPOSITION 2.1.13. If C is of characteristic p, there exists a natural identification $C^\flat \cong C$.

PROOF. The assertion is evident as C is perfect by Proposition 2.1.4. □

2.2. The de Rham period ring B_{dR}

For the rest of this chapter, we write ν for the normalized p-adic valuation on $\mathbb{C}_K$ and $\nu^\flat$ for the valuation on $F = \mathbb{C}_K^\flat$ with $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in F$.

LEMMA 2.2.1. The ring $\mathcal{O}_F$ is a perfect $\mathbb{F}_p$-algebra.

PROOF. The assertion is evident by Proposition 2.1.4 and Proposition 2.1.11. □

Definition 2.2.2. The *infinitesimal period ring* is $A_{\mathrm{inf}} := W(\mathcal{O}_F)$.

Remark. Our definition of A_{inf} relies on Lemma 2.2.1. It is worthwhile to mention that the ring A_{inf} is not $(\mathbb{Q}_p, \Gamma_K)$-regular in any meaningful way.

PROPOSITION 2.2.3. There exists a surjective ring homomorphism $\theta : A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}$ with

$$\theta\left(\sum_{n=0}^{\infty}[c_n]p^n\right) = \sum_{n=0}^{\infty} c_n^\sharp p^n \quad \text{for all } c_n \in \mathcal{O}_F. \tag{2.4}$$

PROOF. Proposition 2.1.9 yields a surjective ring homomorphism $\overline{\theta} : \mathcal{O}_F \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ with $\overline{\theta}(c) = \overline{c^\sharp}$ for each $c \in \mathcal{O}_F$, where $\overline{c^\sharp}$ denotes the image of $c^\sharp$ in $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. Moreover, by construction $\overline{\theta}$ lifts to a multiplicative map $\hat{\theta} : \mathcal{O}_F \to \mathcal{O}_{\mathbb{C}_K}$ with $\hat{\theta}(c) = c^\sharp$ for each $c \in \mathcal{O}_F$. Hence we obtain a ring homomorphism $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_K}$ which satisfies the identity (2.4) by Theorem 2.3.1 in Chapter II.

It remains to establish the surjectivity of θ. Let x be an arbitrary element in $\mathcal{O}_{\mathbb{C}_K}$. Since $\mathcal{O}_{\mathbb{C}_K}$ is p-adically complete, it suffices to find a sequence (c_n) in $\mathcal{O}_F$ with

$$x - \sum_{n=0}^{m} c_n^\sharp p^n \in p^{m+1}\mathcal{O}_{\mathbb{C}_K} \quad \text{for each } m \geq 0.$$

In fact, we can use Lemma 2.1.8 to inductively construct such a sequence by setting each c_m to be an element in $\mathcal{O}_F$ with

$$\frac{1}{p^m}\left(x - \sum_{n=0}^{m-1} c_n^\sharp p^n\right) - c_m^\sharp \in p\mathcal{O}_{\mathbb{C}_K},$$

thereby completing the proof. □

Remark. Our proof remains valid if we replace $\mathbb{C}_K$ by an arbitrary perfectoid field C; in other words, every perfectoid field C yields a surjective ring homomorphism $\theta_C : W(\mathcal{O}_{C^\flat}) \twoheadrightarrow \mathcal{O}_C$.

Definition 2.2.4. We refer to the map θ in Proposition 2.2.3 as the *Fontaine map* and let $\theta[1/p] : A_{\mathrm{inf}}[1/p] \to \mathbb{C}_K$ denote the ring homomorphism induced by θ.

Remark. As explained by Brinon-Conrad [**BC**, Lemma 4.4.1], we can construct the Fontaine map θ without using Theorem 2.3.1 from Chapter II. In this approach, we first define θ as a set theoretic map given by the identity (2.4) and show that θ is indeed a ring homomorphism using descriptions of the ring operations on $A_{\mathrm{inf}} = W(\mathcal{O}_F)$.

PROPOSITION 2.2.5. The ring homomorphism $\theta[1/p] : A_{\mathrm{inf}}[1/p] \to \mathbb{C}_K$ is surjective.

PROOF. For every $c \in \mathbb{C}_K$, there exists an integer $n \geq 0$ with $p^n c \in \mathcal{O}_{\mathbb{C}_K}$. Hence the assertion immediately follows from Proposition 2.2.3. □

Definition 2.2.6. We define the *de Rham local ring* to be

$$B_{\mathrm{dR}}^+ := \varprojlim_i A_{\mathrm{inf}}[1/p]/\ker(\theta[1/p])^i$$

and let $\theta_{\mathrm{dR}}^+ : B_{\mathrm{dR}}^+ \twoheadrightarrow A_{\mathrm{inf}}[1/p]/\ker(\theta[1/p])$ denote the natural projection.

Remark. We will soon define the de Rham period ring B_{dR} to be the fraction field of B_{dR}^+ after verifying that B_{dR}^+ is a discrete valuation ring. At this point, it is instructive to explain Fontaine's insight behind the construction of B_{dR}. As briefly discussed in Chapter I, Fontaine introduced the rings B_{HT} and B_{dR} respectively to formulate the Hodge-Tate decomposition and the de Rham comparison isomorphism. Since the de Rham cohomology admits the Hodge filtration with the Hodge cohomology as its graded vector space, Fontaine aimed to construct B_{dR} as a ring which admits a canonical filtration with B_{HT} as its graded ring. He sought B_{dR} as the fraction field of a complete discrete valuation ring B_{dR}^+ with residue field $\mathbb{C}_K$ so that it admits a filtration $\{ \mathrm{Fil}^n(B_{\mathrm{dR}}) \}_{n\in\mathbb{Z}} := \{ t^n B_{\mathrm{dR}}^+ \}_{n\in\mathbb{Z}}$ for a uniformizer $t \in B_{\mathrm{dR}}^+$ with its graded ring isomorphic to B_{HT}. For a perfect field k of characteristic p, the theory of Witt vectors naturally yields a complete discrete valuation ring with residue field k as noted in Chapter II, Lemma 2.3.8. Fontaine judiciously adjusted the construction of Witt vectors for the field $\mathbb{C}_K$ of characteristic 0 by passing to characteristic p, or by tilting the perfectoid field $\mathbb{C}_K$ in modern language. He began by taking the ring $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ which is evidently of characteristic p. As $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ turns out to be not perfect, Fontaine considered its *perfection* $\varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K} \cong \mathcal{O}_F$ by adding all p-power roots of elements in $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. Fontaine then discovered that $A_{\mathrm{inf}} = W(\mathcal{O}_F)$ gives rise to a surjective ring homomorphism $\theta[1/p] : A_{\mathrm{inf}}[1/p] \twoheadrightarrow \mathbb{C}_K$. Moreover, as we will soon see, $\ker(\theta[1/p])$ turned out to be a principal ideal. Therefore Fontaine obtained the desired ring B_{dR}^+ as the completion of $A_{\mathrm{inf}}[1/p]$ with respect to $\ker(\theta[1/p])$.

Lemma 2.2.7. For each integer $n \geq 0$ we have $\ker(\theta) \cap p^n A_{\mathrm{inf}} = p^n \ker(\theta)$.

Proof. Since we evidently have $p^n \ker(\theta) \subseteq \ker(\theta) \cap p^n A_{\mathrm{inf}}$, we only need to show that every $a \in \ker(\theta) \cap p^n A_{\mathrm{inf}}$ is an element of $p^n \ker(\theta)$. Let us write $a = p^n b$ for some $b \in A_{\mathrm{inf}}$. From the identity

$$0 = \theta(a) = \theta(p^n b) = p^n \theta(b)$$

we find $\theta(b) = 0$ as $\mathcal{O}_{\mathbb{C}_K}$ is torsion free. Therefore we deduce that $a = p^n b$ lies in $p^n \ker(\theta)$ as desired. □

Lemma 2.2.8. The sharp map associated to $\mathbb{C}_K$ is surjective.

Proof. The assertion follows from the fact that $\mathbb{C}_K$ is algebraically closed as noted in Chapter II, Proposition 3.1.13. □

Remark. It is worthwhile to mention that Lemma 2.2.8 is not essential for our discussion. In fact, we use Lemma 2.2.8 only to give a simple description of an element generating $\ker(\theta)$. For an arbitrary perfectoid field C, we can still show that the kernel of the surjective ring homomorphism $\theta_C : W(\mathcal{O}_{C^\flat}) \twoheadrightarrow \mathcal{O}_C$ is principal by explicitly presenting a generator.

Definition 2.2.9. A *distinguished element* of A_{inf} is an element of the form $\xi = [p^\flat] - p \in A_{\mathrm{inf}}$ for some $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\sharp = p$.

Remark. The existence of $p^\flat$ follows from Lemma 2.2.8. We may regard $p^\flat$ as a system of p-power roots of p in $\mathbb{C}_K$.

For the rest of this chapter, we fix a distinguished element $\xi = [p^\flat] - p \in A_{\mathrm{inf}}$.

Lemma 2.2.10. Every element $a \in \ker(\theta)$ is an A_{inf}-linear combination of ξ and p.

Proof. We wish to show that a lies in the ideal generated by ξ and p, or equivalently by $[p^\flat]$ and p. Let us write

$$a = \sum_{n \geq 0} [c_n] p^n = [c_0] + p \sum_{n \geq 1} [c_n] p^{n-1} \qquad \text{with } c_n \in \mathcal{O}_F.$$

It suffices to show that $[c_0]$ is divisible by $[p^\flat]$. Since we have $0 = \theta(a) = \sum\limits_{n \geq 0} c_n^\sharp p^n$, we deduce that $c_0^\sharp$ is divisible by p and consequently find

$$\nu^\flat(c_0) = \nu(c_0^\sharp) \geq \nu(p) = \nu((p^\flat)^\sharp) = \nu^\flat(p^\flat).$$

Hence there exists an element $u \in \mathcal{O}_F$ with $c_0 = p^\flat u$ or equivalently $[c_0] = [p^\flat][u]$. □

Proposition 2.2.11. The element $\xi \in A_{\mathrm{inf}}$ generates the ideal $\ker(\theta)$ in A_{inf}.

Proof. The ideal $\ker(\theta)$ contains ξ as we have

$$\theta(\xi) = \theta([p^\flat] - p) = (p^\flat)^\sharp - p = p - p = 0.$$

Hence we only need to show that every $a \in \ker(\theta)$ lies in the ideal ξA_{inf}. Since A_{inf} is p-adically complete by construction, it suffices to present a sequence (c_n) in A_{inf} with

$$a - \sum_{n=0}^{m} c_n \xi p^n \in p^{m+1} A_{\mathrm{inf}} \qquad \text{for each } m \geq 0.$$

We take $c_0 \in A_{\mathrm{inf}}$ with $a - c_0 \xi \in p A_{\mathrm{inf}}$ given by Lemma 2.2.10 and inductively construct c_m for each $m \geq 1$. In fact, by Lemma 2.2.7 we have

$$a - \sum_{n=0}^{m-1} c_n \xi p^n \in \ker(\theta) \cap p^m A_{\mathrm{inf}} = p^m \ker(\theta)$$

and thus find $b_m, c_m \in A_{\mathrm{inf}}$ with

$$a - \sum_{n=0}^{m-1} c_n \xi p^n = p^m (p b_m + c_m \xi)$$

or equivalently

$$a - \sum_{n=0}^{m} c_n \xi p^n = p^{m+1} b_m$$

as desired. □

Remark. Proposition 2.2.11 yields a natural isomorphism $A_{\mathrm{inf}}/\xi A_{\mathrm{inf}} \cong \mathcal{O}_{\mathbb{C}_K}$, which turns out to be a homeomorphism. Since the construction of A_{inf} depends only on the field F, the principal deal $\xi A_{\mathrm{inf}} \subseteq A_{\mathrm{inf}}$ contains all necessary information for recovering the perfectoid field $\mathbb{C}_K$ from its tilt F. In fact, as we will see in Chapter IV, every perfectoid field C with $C^\flat \simeq F$ arises as the fraction field of A_{inf}/I for a unique principal ideal $I \subseteq A_{\mathrm{inf}}$.

Proposition 2.2.12. The element $\xi \in A_{\mathrm{inf}}$ generates the ideal $\ker(\theta[1/p])$ in $A_{\mathrm{inf}}[1/p]$.

Proof. For every $a \in \ker(\theta[1/p])$, we have $p^n a \in \ker(\theta)$ for some $n > 0$. Hence the assertion follows from Proposition 2.2.11. □

LEMMA 2.2.13. Every $a \in A_{\text{inf}}[1/p]$ with $\xi a \in A_{\text{inf}}$ is an element in A_{inf}.

PROOF. We have $\theta(\xi a) = \theta[1/p](\xi a) = 0$ by Proposition 2.2.12 and in turn find $\xi a \in \xi A_{\text{inf}}$ or equivalently $a \in A_{\text{inf}}$ as A_{inf} is an integral domain. □

LEMMA 2.2.14. For each integer $i \geq 1$, we have $A_{\text{inf}} \cap \ker(\theta[1/p])^i = \ker(\theta)^i$.

PROOF. Since we clearly have $\ker(\theta)^i \subseteq A_{\text{inf}} \cap \ker(\theta[1/p])^i$, we only need to show that every $a \in A_{\text{inf}} \cap \ker(\theta[1/p])^i$ lies in $\ker(\theta)^i$. Proposition 2.2.12 yields an element $b \in A_{\text{inf}}[1/p]$ with $a = \xi^i b$. Hence we find $b \in A_{\text{inf}}$ by Lemma 2.2.13 and consequently deduce the desired assertion from Proposition 2.2.11. □

PROPOSITION 2.2.15. We have $\displaystyle\bigcap_{i=1}^{\infty} \ker(\theta)^i = \bigcap_{i=1}^{\infty} \ker(\theta[1/p])^i = 0$.

PROOF. By Lemma 2.2.14 we have

$$\bigcap_{i=1}^{\infty} \ker(\theta[1/p])^i = \left(\bigcap_{i=1}^{\infty} \ker(\theta)^i\right)[1/p].$$

Hence it suffices to establish the identity $\displaystyle\bigcap_{i=1}^{\infty} \ker(\theta)^i = 0$. Let us take an arbitrary element $c \in \displaystyle\bigcap_{i=1}^{\infty} \ker(\theta)^i$ and write $c = \sum [c_n] p^n$ with $c_n \in \mathcal{O}_F$. Proposition 2.2.11 shows that c is divisible by every power of $\xi = [p^\flat] - p$ in A_{inf}, which in particular implies that c_0 is divisible by every power of $p^\flat$ in $\mathcal{O}_F$. Since we have $\nu^\flat(p^\flat) = \nu((p^\flat)^\sharp) = \nu(p) = 1 > 0$, we find $c_0 = 0$ and in turn write $c = pc'$ for some $c' \in A_{\text{inf}}$. Moreover, Lemma 2.2.14 yields the relation

$$c' \in A_{\text{inf}} \cap \left(\bigcap_{i=1}^{\infty} \ker(\theta)^i\right)[1/p] = A_{\text{inf}} \cap \left(\bigcap_{i=1}^{\infty} \ker(\theta[1/p])^i\right) = \bigcap_{i=1}^{\infty} \ker(\theta)^i.$$

Now a simple induction shows that c is infinitely divisible by p and thus is zero. □

PROPOSITION 2.2.16. The ring B_{dR}^+ is a complete discrete valuation ring with $\ker(\theta_{\text{dR}}^+)$ as the maximal ideal, $\mathbb{C}_K$ as the residue field, and ξ as a uniformizer.

PROOF. Since we have $B_{\text{dR}}^+/\ker(\theta_{\text{dR}}^+) \cong \mathbb{C}_K$ by Proposition 2.2.5, we deduce from ssome general facts stated in the Stacks project [**Sta**, Tag 05GI and Tag 07BH] that B_{dR}^+ is a local ring with $\ker(\theta_{\text{dR}}^+)$ as the maximal ideal and $\mathbb{C}_K$ as the residue field. Let us now consider an arbitrary nonzero element $b \in B_{\text{dR}}^+$. For each integer $i \geq 0$, we write b_i and ξ_i respectively for the images of b and ξ under the projection $B_{\text{dR}}^+ \twoheadrightarrow A_{\text{inf}}[1/p]/\ker(\theta[1/p])^i$. In addition, we take the largest integer $j \geq 0$ with $b_j = 0$. Proposition 2.2.11 implies that for each $i > j$ we may write $b_i = \xi_i^j u_i$ with $u_i \notin \ker(\theta[1/p])/\ker(\theta[1/p])^i$. For each $i > j$ we let u_i' denote the image of u_i in $A_{\text{inf}}[1/p]/\ker(\theta[1/p])^{i-j}$. We observe that the sequence $(u_i')_{i>j}$ depends only on b and gives rise to a unique unit $u \in B_{\text{dR}}^+$ with $b = \xi^j u$. Therefore B_{dR}^+ is a discrete valuation ring with ξ as a uniformizer. Now we deduce from Proposition 2.2.11 and Proposition 2.2.15 that B_{dR}^+ is complete, thereby establishing the desired assertion. □

Remark. Our argument so far in this subsection remains valid if we replace $\mathbb{C}_K$ by an arbitrary algebraically closed perfectoid field of characteristic 0.

Definition 2.2.17. The *de Rham period ring* B_{dR} is the fraction field of B_{dR}^+.

Proposition 2.2.18. Let K_0 denote the fraction field of $W(k)$.

(1) The field K is a finite totally ramified extension of K_0.
(2) There exists a natural commutative diagram

$$\begin{array}{ccc} K_0 & \hookrightarrow & A_{\mathrm{inf}}[1/p] \\ \big\downarrow & & \big\downarrow \\ \overline{K} & \hookrightarrow & B_{\mathrm{dR}}^+ \\ & \searrow & \big\downarrow \theta_{\mathrm{dR}}^+ \\ & & \mathbb{C}_K \end{array}$$

where the diagonal map is the natural inclusion.

Proof. Let us take a uniformizer π of $\mathcal{O}_K$. There exists an integer $e > 0$ with $p = \pi^e u$ for some unit $u \in \mathcal{O}_K$. Hence we obtain a natural ring homomorphism

$$k = \mathcal{O}_K/\pi\mathcal{O}_K \longrightarrow \mathcal{O}_K/\pi^e\mathcal{O}_K = \mathcal{O}_K/p\mathcal{O}_K \tag{2.5}$$

which identifies $\mathcal{O}_K/p\mathcal{O}_K$ as a k-algebra with a basis given by $1, \pi, \cdots, \pi^{e-1}$. The map (2.5) induces a ring homomorphism $W(k) \to \mathcal{O}_K$ by Theorem 2.3.1 in Chapter II.

We assert that $1, \pi, \cdots, \pi^{e-1}$ generate $\mathcal{O}_K$ over $W(k)$. Take an arbitrary element $c \in \mathcal{O}_K$. Since $\mathcal{O}_K$ is p-adically complete, it suffices to find sequences $(a_{0,n}), \cdots, (a_{e-1,n})$ in $W(k)$ with

$$c - \sum_{i=0}^{e-1}\sum_{n=0}^{m} a_{i,n}p^n\pi^i \in p^{m+1}\mathcal{O}_K \qquad \text{for each } m \geq 0.$$

In fact, we use the map (2.5) to inductively obtain $a_{0,m}, \cdots, a_{e-1,m} \in W(k)$ with

$$\frac{1}{p^m}\left(c - \sum_{i=0}^{e-1}\sum_{n=0}^{m-1} a_{i,n}p^n\pi^i\right) - \sum_{i=0}^{e-1} a_{i,m}\pi^i \in p\mathcal{O}_K$$

and consequently obtain the desired assertion.

Our discussion in the previous paragraph shows that K is a finite extension of K_0 and in turn yields statement (1) as both K_0 and K have residue field k. Hence it remains to establish statement (2). The map (2.5) induces a ring homomorphism $k \to \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. Since k is perfect, this map gives rise to a natural homomorphism

$$k \longrightarrow \varprojlim_{x \to x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K} \cong \mathcal{O}_F$$

with the isomorphism given by Proposition 2.1.6 and in turn yields the top horizontal map by Theorem 2.3.1 in Chapter II. Moreover, we get the left vertical map from statement (1) and take the right vertical map to be the natural map

$$A_{\mathrm{inf}}[1/p] \to \varprojlim_{i} A_{\mathrm{inf}}[1/p]/\ker(\theta[1/p])^i = B_{\mathrm{dR}}^+$$

which is injective by Proposition 2.2.15. We may now identify K_0 as a subring of B_{dR}^+. Statement (1) and Proposition 2.2.16 together show that $\overline{K}$ is a separable algebraic extension of K_0 which lies in the residue field $\mathbb{C}_K$ of the complete discrete valuation ring B_{dR}^+. Therefore Hensel's lemma implies that $\overline{K}$ admits a unique embedding into B_{dR}^+ which fits in the desired diagram. $\square$

In order to study some additional properties of B_{dR}, we invoke the following technical result without a proof.

PROPOSITION 2.2.19. There exists a refinement of the discrete valuation topology on B_{dR}^{+} with the following properties:

(i) The natural map $A_{\mathrm{inf}} \to B_{\mathrm{dR}}^{+}$ identifies A_{inf} as a closed subring of B_{dR}^{+}.
(ii) The map $\theta[1/p]$ is continuous and open with respect to the p-adic topology on $\mathbb{C}_K$.
(iii) There exists a continuous map $\log : \mathbb{Z}_p(1) \to B_{\mathrm{dR}}^{+}$ with

$$\log(c) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([c]-1)^n}{n} \qquad \text{for every } c \in \mathbb{Z}_p(1)$$

under the natural identification $\mathbb{Z}_p(1) = \varprojlim \mu_{p^v}(\overline{K}) = \left\{ c \in \mathcal{O}_F : c^\sharp = 1 \right\}$.
(iv) The multiplication by every uniformizer yields a closed embedding on B_{dR}^{+}.
(v) The ring B_{dR}^{+} is complete.

Remark. We will eventually prove Proposition 2.2.19 in Chapter IV after constructing the Fargues-Fontaine curve. There will be no circular reasoning as the construction of the Fargues-Fontaine curve relies only on results that we have discussed prior to Proposition 2.2.19. Readers can find a sketch of the proof in the notes of Brinon-Conrad [**BC**, Exercise 4.5.3].

Let us briefly explain why Proposition 2.2.19 is essential for our discussion. The discrete valuation topology on B_{dR}^{+} has a major defect of not carrying much information about the p-adic topology on $\mathbb{C}_K$. In fact, if we only consider the discrete valuation topology on B_{dR}^{+} the map $\theta[1/p]$ is not continuous with respect to the p-adic topology on $\mathbb{C}_K$. Proposition 2.2.19 allows us to incoorporates the p-adic topology on $\mathbb{C}_K$ in our discussion, which is essential for studying continuous Γ_K-representations.

Definition 2.2.20. We refer to the map $\log : \mathbb{Z}_p(1) \to B_{\mathrm{dR}}^{+}$ given by Proposition 2.2.19 as the *logarithm map* on $\mathbb{Z}_p(1)$.

Remark. In Chapter IV, we will describe the relationship between this logarithm map and the p-adic logarithm $\log_{\mu_{p^\infty}}$.

LEMMA 2.2.21. Let ε be a basis element of $\mathbb{Z}_p(1) = \varprojlim \mu_{p^v}(\overline{K}) = \left\{ c \in \mathcal{O}_F : c^\sharp = 1 \right\}$ over $\mathbb{Z}_p$.

(1) The element ξ divides $[\varepsilon]-1$ in A_{inf}.
(2) We have $\nu^\flat(\varepsilon - 1) = \dfrac{p}{p-1}$.

PROOF. We have $\theta([\varepsilon]-1) = \varepsilon^\sharp - 1 = 1 - 1 = 0$ and thus deduce statement (1) follows from Proposition 2.2.11. Let us now write $\varepsilon = (\zeta_{p^n})$ where each ζ_{p^n} is a primitive p^n-th root of unity in $\overline{K}$. We use Proposition 2.1.7 and the continuity of ν to find

$$\nu^\flat(\varepsilon - 1) = \nu\left((\varepsilon-1)^\sharp\right) = \nu\left(\lim_{n\to\infty}(\zeta_{p^n}-1)^{p^n}\right) = \lim_{n\to\infty} p^n \nu(\zeta_{p^n}-1).$$

The irreducible polynomial of $\zeta_{p^n}-1$ over $\mathbb{Q}_p$ is $f(x) = \sum_{i=0}^{p-1}(x+1)^{ip^{n-1}}$ of degree $p^{n-1}(p-1)$ with constant term p. Hence we have

$$\nu(\zeta_{p^n}-1) = \frac{\nu(p)}{p^{n-1}(p-1)} = \frac{1}{p^{n-1}(p-1)}$$

and consequently establish statement (2). □

PROPOSITION 2.2.22. Let ε be a basis element of $\mathbb{Z}_p(1) = \left\{ c \in \mathcal{O}_F : c^\sharp = 1 \right\}$ over $\mathbb{Z}_p$.

(1) The element $t := \log(\varepsilon) \in B_{\text{dR}}^+$ is a uniformizer.
(2) For every $m \in \mathbb{Z}_p$ we have $\log(\varepsilon^m) = m\log(\varepsilon)$.

PROOF. Let us first consider statement (1). By Proposition 2.2.18 and Lemma 2.2.21, we have $[\varepsilon] - 1 \in \xi A_{\text{inf}}$ and $\dfrac{([\varepsilon]-1)^n}{n} \in \xi^2 B_{\text{dR}}^+$ for each $n \geq 2$. Hence we find

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} \in ([\varepsilon]-1) + \xi^2 B_{\text{dR}}^+.$$

Since ξ is a uniformizer of B_{dR}^+ as noted in Proposition 2.2.16, it suffices to show that $[\varepsilon] - 1$ is not divisible by ξ^2 in B_{dR}^+,

Suppose for contradiction that $[\varepsilon] - 1$ lies in $\xi^2 B_{\text{dR}}^+$. Proposition 2.2.16 shows that $[\varepsilon] - 1$ maps to 0 under the projection $B_{\text{dR}}^+ \twoheadrightarrow A_{\text{inf}}[1/p]/\ker(\theta[1/p])^2$. Hence Proposition 2.2.11 and Lemma 2.2.14 together imply that $[\varepsilon] - 1$ is an element of $\ker(\theta[1/p])^2 \cap A_{\text{inf}} = \xi^2 A_{\text{inf}}$, which means that $[\varepsilon]-1$ is divisible by ξ^2 in A_{inf}. Since the first terms in the Teichmüller expansions for $[\varepsilon] - 1$ and ξ^2 are respectively $[\varepsilon - 1]$ and $[(p^\flat)^2]$, we have

$$\nu^\flat(\varepsilon - 1) \geq \nu^\flat((p^\flat)^2) = 2\nu^\flat(p^\flat) = 2\nu((p^\flat)^\sharp) = 2\nu(p) = 2.$$

If p is odd, we find $\nu^\flat(\varepsilon - 1) < 2$ by Lemma 2.2.21 and in turn obtain a desired contradiction. For $p = 2$, we write $[\varepsilon] - 1 = \xi^2 a$ for some $a \in A_{\text{inf}}$ and compare the coefficients of p in the Teichmüller expansions using Proposition 2.3.6 from Chapter II to obtain the relation $\varepsilon - 1 = c_1^2(p^\flat)^4$ where c_1 denotes the coefficient of p in the Teichmüller expansion of a. Hence for $p = 2$ we have

$$\nu^\flat(\varepsilon - 1) \geq \nu^\flat((p^\flat)^4) = 4\nu^\flat(p^\flat) = 4\nu((p^\flat)^\sharp) = 4\nu(p) = 4$$

and in turn obtain a desired contradiction by Lemma 2.2.21.

It remains to establish statement (2). If m is an integer, we have

$$\log((1+x)^m) = m\log(1+x)$$

as formal power series and thus set $x = \varepsilon - 1$ to find $\log(\varepsilon^m) = m\log(\varepsilon)$. For the general case, let us choose a sequence (m_i) of integers with each $m_i - m$ divisible by p^i. Since $t = \log(\varepsilon)$ is a uniformizer of B_{dR}^+, we find

$$\lim_{i\to\infty} m_i \log(\varepsilon) = m\log(\varepsilon)$$

by Proposition 2.2.19. In addition, it is straightforward to verify the identity

$$\lim_{i\to\infty} \varepsilon^{m_i} = \varepsilon^m$$

with respect to the valuation topology on F. Hence we have

$$\log(\varepsilon^m) = \log\left(\lim_{i\to\infty} \varepsilon^{m_i}\right) = \lim_{i\to\infty} \log(\varepsilon^{m_i}) = \lim_{i\to\infty} m_i \log(\varepsilon) = m\log(\varepsilon)$$

where the second identity follows from the continuity of the logarithm map as noted in Proposition 2.2.19. □

Remark. Statement (2) shows that log is a $\mathbb{Z}_p$-linear homomorphism.

Definition 2.2.23. A *cyclotomic uniformizer* of B_{dR}^+ is an element of the form $t = \log(\varepsilon)$ for some basis element ε of $\mathbb{Z}_p(1)$.

THEOREM 2.2.24 (Fontaine [**Fon82**]). The ring B_{dR} admits a natural action of Γ_K with the following properties:

(i) The logarithm map and θ_{dR}^+ are Γ_K-equivariant.
(ii) Given a cyclotomic uniformizer $t \in B_{\mathrm{dR}}^+$, we have $\gamma(t) = \chi(\gamma)t$ for every $\gamma \in \Gamma_K$.
(iii) Every cyclotomic uniformizer $t \in B_{\mathrm{dR}}^+$ yields a natural Γ_K-equivariant isomorphism
$$\bigoplus_{n\in\mathbb{Z}} t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+ \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_K(n) = B_{\mathrm{HT}}.$$
(iv) B_{dR} is $(\mathbb{Q}_p, \Gamma_K)$-regular with a canonical identification $B_{\mathrm{dR}}^{\Gamma_K} \cong K$.

PROOF. Let us first describe the natural action of Γ_K on B_{dR}. The action of Γ_K on $\mathbb{C}_K$ naturally induces an action on $F = \varprojlim_{x\mapsto x^p} \mathbb{C}_K$ as the p-th power map on $\mathbb{C}_K$ is Γ_K-equivariant. In fact, given an arbitrary element $x = (x_n) \in F$ we have $\gamma(x) = (\gamma(x_n))$ for every $\gamma \in \Gamma_K$. Since $\mathcal{O}_F$ is stable under the action of Γ_K, we apply Theorem 2.3.1 in Chapter II to obtain a natural action of Γ_K on $A_{\mathrm{inf}}[1/p]$ with
$$\gamma\left(\sum [c_n]p^n\right) = \sum [\gamma(c_n)]p^n \quad \text{for each } \gamma \in \Gamma_K \text{ and } c_n \in \mathcal{O}_F.$$
Now we find that θ and $\theta[1/p]$ are both Γ_K-equivariant by construction, whicn in particular implies that both $\ker(\theta)$ and $\ker(\theta[1/p])$ are stable under the action of Γ_K. Hence Γ_K naturally acts on $B_{\mathrm{dR}}^+ = \varprojlim_i A_{\mathrm{inf}}[1/p]/\ker(\theta[1/p])^i$ and its fraction field B_{dR}.

With our discussion in the preceding paragraph, property (i) is straightforward to verify. Moreover, property (i) and Proposition 2.2.22 togther show that every $\gamma \in \Gamma_K$ acts on a cyclotomic uniformizer $t = \log(\varepsilon) \in B_{\mathrm{dR}}^+$ by the relation
$$\gamma(t) = \gamma(\log(\varepsilon)) = \log(\gamma(\varepsilon)) = \log(\varepsilon^{\chi(\gamma)}) = \chi(\gamma)\log(\varepsilon) = \chi(\gamma)t$$
and thus yield property (ii). Now we note by property (i) that the natural isomorphism
$$B_{\mathrm{dR}}^+/tB_{\mathrm{dR}}^+ = B_{\mathrm{dR}}^+/\ker(\theta_{\mathrm{dR}}^+) \cong \mathbb{C}_K$$
is Γ_K-equivariant and in turn obtain a Γ_K-equivariant isomorphism
$$t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+ \simeq \mathbb{C}_K(n) \quad \text{for every } n \in \mathbb{Z}$$
by property (ii) and Lemma 3.1.8 in Chapter II. Since Proposition 2.2.22 implies that a cyclotomic uniformizer of B_{dR}^+ is unique up to $\mathbb{Z}_p^\times$-multiple, we deduce that this isomorphism is canonical and consequently establish property (iii).

It remains to verify property (iv). Example 1.1.2 shows that B_{dR} is $(\mathbb{Q}_p, \Gamma_K)$-regular for being a field extension of $\mathbb{Q}_p$. In addition, property (i) implies that the natural injective homomorphism $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+$ given by Proposition 2.2.18 is Γ_K-equivariant and in turn induces an injective homomorphism
$$K = \overline{K}^{\Gamma_K} \longleftrightarrow (B_{\mathrm{dR}}^+)^{\Gamma_K} \longleftrightarrow B_{\mathrm{dR}}^{\Gamma_K}. \tag{2.6}$$
Now by property (iii) we get an injective K-algebra homomorphism
$$\bigoplus_{n\in\mathbb{Z}} (B_{\mathrm{dR}}^{\Gamma_K} \cap t^n B_{\mathrm{dR}}^+)/(B_{\mathrm{dR}}^{\Gamma_K} \cap t^{n+1}B_{\mathrm{dR}}^+) \longleftrightarrow B_{\mathrm{HT}}^{\Gamma_K}.$$
Since we have $B_{\mathrm{HT}}^{\Gamma_K} \cong K$ by Theorem 3.1.14 in Chapter II, the K-algebra on the source has dimension at most 1. Hence we find $\dim_K B_{\mathrm{dR}}^{\Gamma_K} \leq 1$ and in turn deduce that the map (2.6) is an isomorphism, thereby completing the proof. $\square$

2.3. Filtered vector spaces

In this subsection we set up a categorical framework for our discussion of B_{dR}-admissible representations in the next subsection.

Definition 2.3.1. Let L be an arbitrary field.

(1) A *filtered vector space* over L is a vector space V over L along with a collection of subspaces $\{\operatorname{Fil}^n(V)\}_{n\in\mathbb{Z}}$ that satisfies the following properties:
   (i) $\operatorname{Fil}^n(V) \supseteq \operatorname{Fil}^{n+1}(V)$ for every $n \in \mathbb{Z}$.
   (ii) $\bigcap\limits_{n\in\mathbb{Z}} \operatorname{Fil}^n(V) = 0$ and $\bigcup\limits_{n\in\mathbb{Z}} \operatorname{Fil}^n(V) = V$.

(2) A *graded vector space* over L is a vector space V over L along with a direct sum decomposition $V = \bigoplus\limits_{n\in\mathbb{Z}} V_n$.

(3) A L-linear map between two filtered vector spaces V and W over L is called a *morphism of filtered vector spaces* if it maps each $\operatorname{Fil}^n(V)$ into $\operatorname{Fil}^n(W)$.

(4) A L-linear map between two graded vector spaces $V = \bigoplus\limits_{n\in\mathbb{Z}} V_n$ and $W = \bigoplus\limits_{n\in Z} W_n$ over L is called a *morphism of graded vector spaces* if it maps each V_n into W_n.

(5) For a filtered vector space V over L, we define its *associated graded vector space* by

$$\operatorname{gr}(V) := \bigoplus_{n\in\mathbb{Z}} \operatorname{Fil}^n(V)/\operatorname{Fil}^{n+1}(V)$$

and write $\operatorname{gr}^n(V) := \operatorname{Fil}^n(V)/\operatorname{Fil}^{n+1}(V)$ for every $n \in \mathbb{Z}$.

(6) We denote by Fil_L the category of finite dimensional filtered vector spaces over L.

Example 2.3.2. We present some motivating examples for our discussion.

(1) Theorem 2.2.24 shows that B_{dR} is a filtered K-algebra with $\operatorname{Fil}^n(B_{\mathrm{dR}}) := t^n B_{\mathrm{dR}}^+$ and $\operatorname{gr}(B_{\mathrm{dR}}) \cong B_{\mathrm{HT}}$ where t is a cyclotomic uniformizer of B_{dR}^+.

(2) For a proper smooth variety X over K, the de Rham cohomology $H^n_{\mathrm{dR}}(X/K)$ with the Hodge filtration is a filtered vector space over K whose associated graded vector space recovers the Hodge cohomology.

(3) For every $V \in \operatorname{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, we may regard $D_{B_{\mathrm{dR}}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K}$ as a filtered vector space over K with

$$\operatorname{Fil}^n(D_{B_{\mathrm{dR}}}(V)) := (V \otimes_{\mathbb{Q}_p} t^n B_{\mathrm{dR}}^+)^{\Gamma_K}.$$

Remark. For an arbitrary proper smooth variety X over K, we have a canonical Γ_K-equivariant isomorphism of filtered vector spaces

$$D_{B_{\mathrm{dR}}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^n_{\mathrm{dR}}(X/K)$$

by Theorem 1.2.3 in Chapter I. In particular, we can recover the Hodge filtration on $H^n_{\mathrm{dR}}(X/K)$ from the Γ_K-action on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$.

Lemma 2.3.3. Let V be a finite dimensional filtered vector space over a field L. There exists a basis $(v_{i,j})$ for V such that for every $n \in \mathbb{Z}$ the vectors $v_{i,j}$ with $i \geq n$ form a basis for $\operatorname{Fil}^n(V)$.

Proof. Since V is finite dimensional, we have $\operatorname{Fil}^n(V) = 0$ for all sufficiently large n and $\operatorname{Fil}^n(V) = 0$ for all sufficiently small n. Hence we can construct such a basis by inductively extending a basis for $\operatorname{Fil}^n(V)$ to a basis for $\operatorname{Fil}^{n-1}(V)$. □

Definition 2.3.4. Let L be an arbitrary field.

(1) Given two filtered vector spaces V and W over L, we define the *convolution filtration* on $V \otimes_L W$ by
$$\mathrm{Fil}^n(V \otimes_L W) := \sum_{i+j=n} \mathrm{Fil}^i(V) \otimes_L \mathrm{Fil}^j(W).$$

(2) For every filtered vector space V over L, we define the *dual filtration* on the dual space $V^\vee = \mathrm{Hom}_L(V, L)$ by
$$\mathrm{Fil}^n(V^\vee) := \left\{\, f \in V^\vee : \mathrm{Fil}^{1-n}(V) \subseteq \ker(f) \,\right\}.$$

(3) We define the *unit object* $L[0]$ in Fil_L to be the vector space L with the filtration
$$\mathrm{Fil}^n(L[0]) := \begin{cases} L & \text{if } n \leq 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Remark. The use of $\mathrm{Fil}^{1-n}(V)$ rather than $\mathrm{Fil}^{-n}(V)$ in (2) is to ensure that $L[0]$ is self-dual.

Proposition 2.3.5. Let V be a filtered vector space over a field L. Then we have canonical isomorphisms of filtered vector spaces
$$V \otimes_L L[0] \cong L[0] \otimes_L V \cong V \qquad \text{and} \qquad (V^\vee)^\vee \cong V.$$

Proof. For every $n \in \mathbb{Z}$ we find
$$\mathrm{Fil}^n(V \otimes_L L[0]) = \sum_{i+j=n} \mathrm{Fil}^i(V) \otimes_L \mathrm{Fil}^j(L[0]) \cong \sum_{i \geq n} \mathrm{Fil}^i(V) = \mathrm{Fil}^n(V),$$
and consequently obtain an identification of filtered vector spaces
$$V \otimes_L L[0] \cong L[0] \otimes_L V \cong V.$$
Moreover, the natural evaluation isomorphism $\mathfrak{e} : V \cong (V^\vee)^\vee$ yields an isomorphism of filtered vector spaces since for every $n \in \mathbb{Z}$ we have
$$\begin{aligned} \mathrm{Fil}^n\left((V^\vee)^\vee\right) &\cong \left\{\, v \in V : \mathrm{Fil}^{1-n}(V^\vee) \subseteq \ker(\mathfrak{e}(v)) \,\right\} \\ &= \left\{\, v \in V : f(v) = 0 \text{ for all } f \in \mathrm{Fil}^{1-n}(V^\vee) \,\right\} \\ &= \left\{\, v \in V : f(v) = 0 \text{ for all } f \in V^\vee \text{ with } \mathrm{Fil}^n(V) \subseteq \ker(f) \,\right\} \\ &= \mathrm{Fil}^n(V). \end{aligned}$$
Therefore we complete the proof. □

Proposition 2.3.6. Let V and W be finite dimensional filtered vector spaces over a field L. Then we have a natural identification of filtered vector spaces
$$(V \otimes_L W)^\vee \cong V^\vee \otimes_L W^\vee.$$

Proof. By Lemma 2.3.3 we can choose bases $(v_{i,k})$ and $(w_{j,l})$ for V and W such that for every $n \in \mathbb{Z}$ the vectors $(v_{i,k})_{i \geq n}$ and $(w_{j,l})_{j \geq n}$ respectively form bases for $\mathrm{Fil}^n(V)$ and $\mathrm{Fil}^n(W)$. Let $(f_{i,k})$ and $(g_{j,l})$ be the dual bases for $V^\vee$ and $W^\vee$. Then the vectors $(f_{i,k} \otimes g_{j,l})$ form a basis for the vector space $(V \otimes_L W)^\vee \cong V^\vee \otimes_L W^\vee$. Moreover, for every $n \in \mathbb{Z}$ the vectors $(f_{i,k})_{i \leq -n}$ and $(g_{j,l})_{j \leq -n}$ respectively form bases for $\mathrm{Fil}^n(V^\vee)$ and $\mathrm{Fil}^n(W^\vee)$. Hence we find that for every $n \in \mathbb{Z}$ both $\mathrm{Fil}^n\left((V \otimes_L W)^\vee\right)$ and $\mathrm{Fil}^n(V^\vee \otimes_L W^\vee)$ are spanned by the vectors $(f_{i,k} \otimes g_{j,l})_{i+j \leq -n}$, thereby deducing the desired assertion. □

LEMMA 2.3.7. Let $V = \bigoplus_{n\in\mathbb{Z}} V_n$ and $W = \bigoplus_{n\in\mathbb{Z}} W_n$ be graded vector spaces over a field L. A morphism $f : V \longrightarrow W$ of graded vector spaces is an isomorphism if and only if it is bijective.

PROOF. The assertion immediately follows by observing that f is the direct sum of the induced morphisms $f_n : V_n \longrightarrow W_n$. □

PROPOSITION 2.3.8. Let L be an arbitrary field. A bijective morphism $f : V \longrightarrow W$ in Fil_L is an isomorphism in Fil_L if and only if the induced map $\mathrm{gr}(f) : \mathrm{gr}(V) \longrightarrow \mathrm{gr}(W)$ is bijective.

PROOF. If f is an isomorphism of filtered vector spaces, then $\mathrm{gr}(f)$ is clearly an isomorphism. Let us now assume that $\mathrm{gr}(f)$ is an isomorphism. We wish to show that for every $n \in \mathbb{Z}$ the induced map $\mathrm{Fil}^n(f) : \mathrm{Fil}^n(V) \longrightarrow \mathrm{Fil}^n(W)$ is an isomorphism. Since each $\mathrm{Fil}^n(f)$ is injective by the bijectivity of f, it suffices to show

$$\dim_L \mathrm{Fil}^n(V) = \dim_L \mathrm{Fil}^n(W) \qquad \text{for every } n \in \mathbb{Z}.$$

The map $\mathrm{gr}(f)$ is an isomorphism of graded vector spaces by Lemma 2.3.7, and consequently induces an isomorphism

$$\mathrm{gr}^n(V) \simeq \mathrm{gr}^n(W) \qquad \text{for every } n \in \mathbb{Z}.$$

Hence for every $n \in \mathbb{Z}$ we find

$$\dim_L \mathrm{Fil}^n(V) = \sum_{i\geq n} \dim_L \mathrm{gr}^i(V) = \sum_{i\geq n} \dim_L \mathrm{gr}^i(W) = \dim_L \mathrm{Fil}^n(W)$$

as desired. □

Example 2.3.9. Let us define $L[1]$ to be the vector space L with the filtration

$$\mathrm{Fil}^n(L[1]) := \begin{cases} L & \text{if } n \leq 1, \\ 0 & \text{if } n > 1. \end{cases}$$

The bijective morphism $L[0] \longrightarrow L[1]$ given by the identity map on L is not an isomorphism in Fil_L since $\mathrm{Fil}^1(L[0]) = 0$ and $\mathrm{Fil}^1(L[1]) = L$ are not isomorphic. Moreover, the induced map $\mathrm{gr}(L[0]) \longrightarrow \mathrm{gr}(L[1])$ is a zero map.

PROPOSITION 2.3.10. Let L be an arbitrary field. For any $V, W \in \mathrm{Fil}_L$ there exists a natural isomorphism of graded vector spaces

$$\mathrm{gr}(V \otimes_L W) \cong \mathrm{gr}(V) \otimes_L \mathrm{gr}(W).$$

PROOF. Since we have a direct sum decomposition

$$\mathrm{gr}(V) \otimes_L \mathrm{gr}(W) = \bigoplus_{n\in\mathbb{Z}} \left(\bigoplus_{i+j=n} \mathrm{gr}^i(V) \otimes_L \mathrm{gr}^j(W) \right),$$

it suffices to find a natural isomorphism

$$\mathrm{gr}^n(V \otimes_L W) \cong \bigoplus_{i+j=n} \mathrm{gr}^i(V) \otimes_L \mathrm{gr}^j(W) \qquad \text{for every } n \in \mathbb{Z}. \tag{2.7}$$

By Lemma 2.3.3 we can choose bases $(v_{i,k})$ and $(w_{j,l})$ for V and W such that for every $n \in \mathbb{Z}$ the vectors $(v_{i,k})_{i\geq n}$ and $(w_{j,l})_{j\geq n}$ respectively span $\mathrm{Fil}^n(V)$ and $\mathrm{Fil}^n(W)$. Let $\overline{v}_{i,k}$ denote the image of $v_{i,k}$ under the map $\mathrm{Fil}^i(V) \twoheadrightarrow \mathrm{gr}^i(V)$, and let $\overline{w}_{j,l}$ denote the image of $w_{j,l}$ under the map $\mathrm{Fil}^j(W) \twoheadrightarrow \mathrm{gr}^j(W)$. Since each $\mathrm{Fil}^n(V \otimes_L W)$ is spanned by the vectors $(v_{i,k} \otimes w_{j,l})_{i+j\geq n}$, we obtain the identification (2.7) by observing that both sides are spanned by the vectors $(\overline{v}_{i,k} \otimes \overline{w}_{j,l})_{i+j=n}$. □

2.4. Properties of de Rham representations

Definition 2.4.1. We say that $V \in \text{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is *de Rham* if it is B_{dR}-admissible. We write $\text{Rep}_{\mathbb{Q}_p}^{\text{dR}}(\Gamma_K) := \text{Rep}_{\mathbb{Q}_p}^{B_{\text{dR}}}(\Gamma_K)$ for the category of de Rham p-adic Γ_K-representations. In addition, we write D_{HT} and D_{dR} respectively for the functors $D_{B_{\text{HT}}}$ and $D_{B_{\text{dR}}}$.

Example 2.4.2. Below are some important examples of de Rham representations.

(1) For every $n \in \mathbb{Z}$ the Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ is de Rham; indeed, the inequality
$$\dim_K D_{\text{dR}}(\mathbb{Q}_p(n)) \leq \dim_{\mathbb{Q}_p} \mathbb{Q}_p(n) = 1$$
given by Theorem 1.2.1 is an equality, as $D_{\text{dR}}(\mathbb{Q}_p(n)) = (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\text{dR}})^{\Gamma_K}$ contains a nonzero element $1 \otimes t^{-n}$ by Theorem 2.2.24.

(2) Every $\mathbb{C}_K$-admissible representation is de Rham by a result of Sen.

(3) For every proper smooth variety X over K, the étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is de Rham by a theorem of Faltings as briefly discussed in Chapter I, Theorem 1.2.3.

The general formalism discussed in §1 readily yields a number of nice properties for de Rham representations and the functor D_{dR}. Our main goal in this subsection is to extend these properties in order to incorporate the additional structures induced by the filtration $\{ t^n B_{\text{dR}}^+ \}_{n \in \mathbb{Z}}$ on B_{dR}.

Lemma 2.4.3. Given any $n \in \mathbb{Z}$, every $V \in \text{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is de Rham if and only if $V(n)$ is de Rham.

Proof. Since we have identifications
$$V(n) \cong V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n) \qquad \text{and} \qquad V \cong V(n) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-n),$$
the assertion follows from Proposition 1.2.4 and the fact that every Tate twist of $\mathbb{Q}_p$ is de Rham as noted in Example 2.4.2. □

Proposition 2.4.4. Let V be a de Rham representation of Γ_K. Then V is Hodge-Tate with a natural K-linear isomorphism of graded vector spaces
$$\text{gr}(D_{\text{dR}}(V)) \cong D_{\text{HT}}(V).$$

Proof. For every $n \in \mathbb{Z}$ we have a short exact sequence
$$0 \longrightarrow t^{n+1}B_{\text{dR}}^+ \longrightarrow t^n B_{\text{dR}}^+ \longrightarrow t^n B_{\text{dR}}^+/t^{n+1}B_{\text{dR}}^+ \longrightarrow 0,$$
which induces an exact sequence
$$0 \longrightarrow \left(V \otimes_{\mathbb{Q}_p} t^{n+1}B_{\text{dR}}^+\right)^{\Gamma_K} \longrightarrow \left(V \otimes_{\mathbb{Q}_p} t^n B_{\text{dR}}^+\right)^{\Gamma_K} \longrightarrow \left(V \otimes_{\mathbb{Q}_p} (t^n B_{\text{dR}}^+/t^{n+1}B_{\text{dR}}^+)\right)^{\Gamma_K}$$
and consequently yields an injective K-linear map
$$\text{gr}^n(D_{\text{dR}}(V)) = \text{Fil}^n(D_{\text{dR}}(V))/\,\text{Fil}^{n+1}(D_{\text{dR}}(V)) \hookrightarrow \left(V \otimes_{\mathbb{Q}_p} (t^n B_{\text{dR}}^+/t^{n+1}B_{\text{dR}}^+)\right)^{\Gamma_K}.$$
Therefore we obtain an injective K-linear map of graded vector spaces
$$\text{gr}(D_{\text{dR}}(V)) \hookrightarrow \bigoplus_{n \in \mathbb{Z}} \left(V \otimes_{\mathbb{Q}_p} (t^n B_{\text{dR}}^+/t^{n+1}B_{\text{dR}}^+)\right)^{\Gamma_K} \cong (V \otimes_{\mathbb{Q}_p} B_{\text{HT}})^{\Gamma_K} = D_{\text{HT}}(V)$$
where the middle isomorphism follows from Theorem 2.2.24. We then find
$$\dim_K D_{\text{dR}}(V) = \dim_K \text{gr}(D_{\text{dR}}(V)) \leq \dim_K D_{\text{HT}}(V) \leq \dim_{\mathbb{Q}_p} V$$
where the last inequality follows from Theorem 1.2.1. Since V is de Rham, both inequalities should be in fact equalities, thereby yielding the desired assertion. □

Example 2.4.5. Let V be an extension of $\mathbb{Q}_p(m)$ by $\mathbb{Q}_p(n)$ with $m < n$. We assert that V is de Rham. By Lemma 2.4.3 we may assume $m = 0$. Then we have a short exact sequence

$$0 \longrightarrow \mathbb{Q}_p(n) \longrightarrow V \longrightarrow \mathbb{Q}_p \longrightarrow 0. \tag{2.8}$$

Since the functor D_{dR} is left exact by construction, we obtain a left exact sequence

$$0 \longrightarrow D_{\mathrm{dR}}(\mathbb{Q}_p(n)) \longrightarrow D_{\mathrm{dR}}(V) \longrightarrow D_{\mathrm{dR}}(\mathbb{Q}_p).$$

We wish to show $\dim_K D_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V = 2$. Since we have

$$\dim_K D_{\mathrm{dR}}(\mathbb{Q}_p(n)) = \dim_K D_{\mathrm{dR}}(\mathbb{Q}_p) = 1$$

by Example 2.4.2, it suffices to show the surjectivity of the map $D_{\mathrm{dR}}(V) \longrightarrow D_{\mathrm{dR}}(\mathbb{Q}_p) \cong K$.

As B_{dR}^+ is faithfully flat over $\mathbb{Q}_p$, the sequence (2.8) yields a short exact sequence

$$0 \longrightarrow \mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+ \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+ \longrightarrow \mathbb{Q}_p \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+ \longrightarrow 0.$$

In addition, by Theorem 2.2.24 and Proposition 2.2.18 we have identifications

$$(\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\Gamma_K} \cong (t^n B_{\mathrm{dR}}^+)^{\Gamma_K} = 0,$$
$$(\mathbb{Q}_p \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\Gamma_K} \cong (B_{\mathrm{dR}}^+)^{\Gamma_K} \cong K.$$

We thus obtain a long exact sequence

$$0 \longrightarrow 0 \longrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\Gamma_K} \longrightarrow K \longrightarrow H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+).$$

Since we have $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\Gamma_K} \subseteq (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K} = D_{\mathrm{dR}}(V)$, it suffices to prove

$$H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+) = 0. \tag{2.9}$$

By Theorem 2.2.24 we have a short exact sequence

$$0 \longrightarrow t^{n+1} B_{\mathrm{dR}}^+ \longrightarrow t^n B_{\mathrm{dR}}^+ \longrightarrow \mathbb{C}_K(n) \longrightarrow 0,$$

which in turn yields a long exact sequence

$$\mathbb{C}_K(n)^{\Gamma_K} \longrightarrow H^1(\Gamma_K, t^{n+1} B_{\mathrm{dR}}^+) \longrightarrow H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+) \longrightarrow H^1(\Gamma_K, \mathbb{C}_K(n)).$$

Then by Theorem 3.1.14 in Chapter II we obtain an identification

$$H^1(\Gamma_K, t^{n+1} B_{\mathrm{dR}}^+) \cong H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+). \tag{2.10}$$

Hence by induction we only need to prove (2.9) for $n = 1$.

Take an arbitrary element $\alpha_1 \in H^1(\Gamma_K, t B_{\mathrm{dR}}^+)$. We wish to show $\alpha_1 = 0$. Regarding α_1 as a cocycle, we use (2.10) to inductively construct sequences (α_m) and (y_m) with the following properties:

(i) $\alpha_m \in H^1(\Gamma_K, t^m B_{\mathrm{dR}}^+)$ and $y_m \in t^m B_{\mathrm{dR}}^+$ for all $m \geq 1$,

(ii) $\alpha_{m+1}(\gamma) = \alpha_m(\gamma) + \gamma(y_m) - y_m$ for all $\gamma \in \Gamma_K$ and $m \geq 1$.

Now, since t is a uniformizer in B_{dR}^+ as noted in Proposition 2.2.22, we may take an element $y = \sum y_m \in B_{\mathrm{dR}}^+$. Then we have

$$\alpha_1(\gamma) + \gamma(y) - y \in H^1(\Gamma_K, t^m B_{\mathrm{dR}}^+) \qquad \text{for all } \gamma \in \Gamma_K \text{ and } m \geq 0,$$

and consequently find $\alpha_1(\gamma) + \gamma(y) - y = 0$ for all $\gamma \in \Gamma_K$. We thus deduce $\alpha_1 = 0$ as desired.

Remark. It is a highly nontrivial fact that every non-splitting extension of $\mathbb{Q}_p(1)$ by $\mathbb{Q}_p$ in $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is Hodge-Tate but not de Rham. The existence of such an extension follows from the identification

$$\mathrm{Ext}^1_{\mathbb{Q}_p[\Gamma_K]}(\mathbb{Q}_p(1), \mathbb{Q}_p) \cong H^1(\Gamma_K, \mathbb{Q}_p(-1)) \cong K$$

where the second isomorphism is a consequence of the Tate local duality for p-adic representations. Moreover, such an extension is Hodge-Tate as noted in Example 1.1.12. The difficult part is to prove that such an extension is not de Rham. For this part we need a very deep result that every de Rham representation is *potentially semistable*.

PROPOSITION 2.4.6. Let V be a de Rham representation of Γ_K. For every $n \in \mathbb{Z}$ we have $\mathrm{gr}^n(D_{\mathrm{dR}}(V)) \neq 0$ if and only if n is a Hodge-Tate weight of V.

PROOF. This is an immediate consequence of Proposition 2.4.4 and Definition 1.1.14. □

Remark. Proposition 2.4.6 provides the main reason for our choice of the sign convention in the definition of Hodge-Tate weights. In fact, under our convention the Hodge-Tate weights of a de Rham representation V indicate where the filtration of $D_{\mathrm{dR}}(V)$ has a jump. In particular, for a proper smooth variety X over K, the Hodge-Tate weights of the étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ give the positions of "jumps" for the Hodge filtration on the de Rham cohomology $H^n_{\mathrm{dR}}(X/K)$ by the isomorphism of filtered vector spaces

$$D_{\mathrm{dR}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^n_{\mathrm{dR}}(X/K).$$

Example 2.4.7. The Tate twist $\mathbb{Q}_p(m)$ of $\mathbb{Q}_p$ is a 1-dimensional de Rham representation with the Hodge-Tate weight $-m$ as noted in Example 1.1.15 and Example 2.4.2. Hence by Proposition 2.4.6 we find

$$\mathrm{Fil}^n(D_{\mathrm{dR}}(\mathbb{Q}_p(m))) \cong \begin{cases} K & \text{for } n \leq -m, \\ 0 & \text{for } n > -m. \end{cases}$$

In particular, for $m = 0$ we obtain an identification $D_{\mathrm{dR}}(\mathbb{Q}_p) \cong K[0]$.

PROPOSITION 2.4.8. For every $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\Gamma_K)$, we have a natural Γ_K-equivariant isomorphism of filtered vector spaces

$$D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}.$$

PROOF. Since V is de Rham, Theorem 1.2.1 implies that the natural map

$$D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \longrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) \otimes_K B_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{dR}} \otimes_K B_{\mathrm{dR}}) \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

is a Γ_K-equivariant isomorphism of vector spaces over B_{dR}. Moreover, this map is a morphism of filtered vector spaces as each arrow above is easily seen to be a morphism of filtered vector spaces. Hence by Proposition 2.3.8 it suffices to show that the induced map

$$\mathrm{gr}(D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}) \longrightarrow \mathrm{gr}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) \tag{2.11}$$

is an isomorphism. By Proposition 2.3.10, Proposition 2.4.4 and Theorem 2.2.24 we obtain identifications

$$\mathrm{gr}(D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}) \cong \mathrm{gr}(D_{\mathrm{dR}}(V)) \otimes_K \mathrm{gr}(B_{\mathrm{dR}}) \cong D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}},$$
$$\mathrm{gr}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) \cong V \otimes_{\mathbb{Q}_p} \mathrm{gr}(B_{\mathrm{dR}}) \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}.$$

We thus identify the map (2.11) with the natural map

$$D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}} \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}$$

given by Theorem 1.2.1. The desired assertion now follows by Proposition 2.4.4. □

Proposition 2.4.9. The functor D_{dR} with values in Fil_K is faithful and exact on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$.

Proof. Let Vect_K denote the category of finite dimensional vector spaces over K. The faithfulness of D_{dR} on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ is an immediate consequence of Proposition 1.2.2 since the forgetful functor $\mathrm{Fil}_K \longrightarrow \mathrm{Vect}_K$ is faithful. Hence it remains to verify the exactness of D_{dR} on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$. Consider an exact sequence of de Rham representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0. \tag{2.12}$$

The functor D_{dR} with values in Fil_K is left exact by construction. In other words, for every $n \in \mathbb{Z}$ we have a left exact sequence

$$0 \longrightarrow \mathrm{Fil}^n(D_{\mathrm{dR}}(U)) \longrightarrow \mathrm{Fil}^n(D_{\mathrm{dR}}(V)) \longrightarrow \mathrm{Fil}^n(D_{\mathrm{dR}}(W)). \tag{2.13}$$

We wish to show that this sequence extends to a short exact sequence. By Proposition 1.2.2 the sequence (2.12) induces a short exact sequence of vector spaces

$$0 \longrightarrow D_{\mathrm{HT}}(U) \longrightarrow D_{\mathrm{HT}}(V) \longrightarrow D_{\mathrm{HT}}(W) \longrightarrow 0.$$

Moreover, by the definition of D_{HT} we find that this sequence is indeed a short exact sequence of graded vector spaces. Then by Proposition 2.4.4 we may rewrite this sequence as

$$0 \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(U)) \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(V)) \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(W)) \longrightarrow 0.$$

by Proposition 2.4.4. Hence for every $n \in \mathbb{Z}$ we have

$$\begin{aligned}\dim_K \mathrm{Fil}^n(D_{\mathrm{dR}}(V)) &= \sum_{i \geq n} \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V)) \\ &= \sum_{i \geq n} \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(U)) + \sum_{i \geq n} \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(W)) \\ &= \dim_K \mathrm{Fil}^n(D_{\mathrm{dR}}(U)) + \dim_K \mathrm{Fil}^n(D_{\mathrm{dR}}(W)),\end{aligned}$$

thereby deducing that the sequence (2.13) extends to a short exact sequence as desired. □

Corollary 2.4.10. Let V be a de Rham representation. Every subquotient W of V is a de Rham representation with $D_{\mathrm{dR}}(W)$ naturally identified as a subquotient of $D_{\mathrm{dR}}(V)$ in Fil_K.

Proof. This is an immediate consequence of Proposition 1.2.3 and Proposition 2.4.9. □

Proposition 2.4.11. Given any $V, W \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$, we have $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ with a natural isomorphism of filtered vector spaces

$$D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(W) \cong D_{\mathrm{dR}}(V \otimes_{\mathbb{Q}_p} W). \tag{2.14}$$

Proof. By Proposition 1.2.4 we find $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ and obtain the desired isomorphism (2.14) as a map of vector spaces. Moreover, since the construction of the map (2.14) rests on the multiplicative structure of B_{dR} as shown in the proof of Proposition 1.2.4, it is straightforward to verify that the map (2.14) is a morphism in Fil_K. Hence by Proposition 2.3.8 it suffices to show that the induced map

$$\mathrm{gr}(D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(W)) \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(V \otimes_{\mathbb{Q}_p} W)) \tag{2.15}$$

is an isomorphism. Since both V and W are Hodge-Tate by Proposition 2.4.4, we have a natural isomorphism

$$D_{\mathrm{HT}}(V) \otimes_K D_{\mathrm{HT}}(W) \cong D_{\mathrm{HT}}(V \otimes_{\mathbb{Q}_p} W) \tag{2.16}$$

by Proposition 1.2.4. Therefore we complete the proof by identifying the maps (2.15) and (2.16) using Proposition 2.3.10 and Proposition 2.4.4. □

Proposition 2.4.12. For every de Rham representation V, we have $\wedge^n(V) \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ and $\mathrm{Sym}^n(V) \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ with natural isomorphisms of filtered vector spaces

$$\wedge^n(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{dR}}(\wedge^n(V)) \qquad \text{and} \qquad \mathrm{Sym}^n(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{dR}}(\mathrm{Sym}^n(V)).$$

Proof. Proposition 1.2.5 implies that both $\wedge^n(V)$ and $\mathrm{Sym}^n(V)$ are de Rham for every $n \geq 1$. In addition, Proposition 1.2.5 yields the desired isomorphisms as maps of vector spaces. Then Corollary 2.4.10 and Proposition 2.4.11 together imply that these maps are isomorphisms in Fil_K. ☐

Proposition 2.4.13. For every de Rham representation V, the dual representation $V^\vee$ is de Rham with a natural perfect paring of filtered vector spaces

$$D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(V^\vee) \cong D_{\mathrm{dR}}(V \otimes_{\mathbb{Q}_p} V^\vee) \longrightarrow D_{\mathrm{dR}}(\mathbb{Q}_p) \cong K[0]. \tag{2.17}$$

Proof. By Proposition 1.2.6 we find $V^\vee \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ and obtain the desired perfect pairing as a map of vector spaces. Moreover, Proposition 2.4.11 implies that this pairing is a morphism in Fil_K. We thus obtain a bijective morphism of filtered vector spaces

$$D_{\mathrm{dR}}(V)^\vee \longrightarrow D_{\mathrm{dR}}(V^\vee).$$

Therefore by Proposition 2.3.8 it suffices to show that the induced map

$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(V^\vee)) \tag{2.18}$$

is an isomorphism. Since V is Hodge-Tate by Proposition 2.4.4, we have a natural isomorphism

$$D_{\mathrm{HT}}(V)^\vee \cong D_{\mathrm{HT}}(V^\vee) \tag{2.19}$$

by Proposition 1.2.6. We thus deduce the desired assertion by identifying the maps (2.18) and (2.19) using Proposition 2.4.4. ☐

Let us now discuss some additional facts about de Rham representations and the functor D_{dR}.

Proposition 2.4.14. Let V be a p-adic representation of Γ_K. Let L be a finite extension of K with absolute Galois group Γ_L.

(1) There exists a natural isomorphism of filtered vector spaces

$$D_{\mathrm{dR},K}(V) \otimes_K L \cong D_{\mathrm{dR},L}(V)$$

where we set $D_{\mathrm{dR},K}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K}$ and $D_{\mathrm{dR},L}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_L}$.

(2) V is de Rham if and only if it is de Rham as a representation of Γ_L.

Proof. We only need to prove the first statement, as the second statement immediately follows from the first statement. Let L' be the Galois closure of L over K with the absolute Galois group $\Gamma_{L'}$ and set $D_{\mathrm{dR},L'}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_{L'}}$. Then we have identifications

$$D_{\mathrm{dR},K}(V) \otimes_K L = (D_{\mathrm{dR},K}(V) \otimes_K L')^{\mathrm{Gal}(L'/L)} \qquad \text{and} \qquad D_{\mathrm{dR},L}(V) = D_{\mathrm{dR},L'}(V)^{\mathrm{Gal}(L'/L)}.$$

Hence we may replace L by L' to assume that L is Galois over K. Moreover, since the construction of B_{dR} depends only on $\mathbb{C}_K$, we get a natural L-linear map

$$D_{\mathrm{dR},K}(V) \otimes_K L \longrightarrow D_{\mathrm{dR},L}(V).$$

It is evident that this map induces a morphism of filtered vector spaces over L where the filtrations on the source and the target are given as in Example 2.4.2. We then have

$$\mathrm{Fil}^n(D_{\mathrm{dR},K}(V)) = \mathrm{Fil}^n(D_{\mathrm{dR},L}(V))^{\mathrm{Gal}(L/K)} \qquad \text{for all } n \in \mathbb{Z},$$

thereby deducing the desired assertion by the Galois descent for vector spaces. ☐

Remark. Proposition 2.4.14 extends to any complete discrete-valued extension L of K inside $\mathbb{C}_K$, based on the "completed unramified descent argument" as explained in [**BC**, Proposition 6.3.8]. This fact has the following immediate consequences:

(1) Every potentially unramified p-adic representation is de Rham; indeed, we have already mentioned this in Example 2.4.2 since being $\mathbb{C}_K$-admissible is the same as being potentially unramified as noted in Example 1.1.4.
(2) For one-dimensional p-adic representations, being de Rham is the same as being Hodge-Tate by Proposition 1.1.13 and Lemma 2.4.3.

Example 2.4.15. Let $\eta : \Gamma_K \longrightarrow \mathbb{Z}_p^\times$ be a continuous character with finite image. Then there exists a finite extension L of K with absolute Galois group Γ_L such that $\mathbb{Q}_p(\eta)$ is trivial as a representation of Γ_L. Hence by Example 2.4.7 and Proposition 2.4.14 we find that $\mathbb{Q}_p(\eta)$ is de Rham with an isomorphism of filtered vector spaces

$$D_{\mathrm{dR}}(\mathbb{Q}_p(\eta)) \otimes_K L \cong L[0],$$

and consequently obtain an identification

$$D_{\mathrm{dR}}(\mathbb{Q}_p(\eta)) \cong K[0] \cong D_{\mathrm{dR}}(\mathbb{Q}_p).$$

In particular, we deduce that the functor D_{dR} on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ with values in Fil_K is not full.

We close this section by introducing a very important conjecture, known as the *Fontaine-Mazur conjecture*, which predicts a criterion for the "geometricity" of global p-adic representations.

Conjecture 2.4.16 (Fontaine-Mazur [**FM95**]). Fix a number field E, and denote by $\mathcal{O}_E$ the ring of integers in E. Let V be a finite dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ over $\mathbb{Q}_p$ with the following properties:

(i) For all but finitely many prime ideals $\mathfrak{p}$ of $\mathcal{O}_E$, the representation V is unramified at $\mathfrak{p}$ in the sense that the action of the inertia group at $\mathfrak{p}$ is trivial.
(ii) For all prime ideals of $\mathcal{O}_E$ lying over p, the restriction of V to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E_\mathfrak{p})$ is de Rham.

Then there exist a proper smooth variety X over E such that V appears as a subquotient of the étale cohomology $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(m))$ for some $m, n \in \mathbb{Z}$.

Remark. If V is one-dimensional, then Conjecture 2.4.16 follows essentially by the class field theory. For two-dimensional representations, Conjecture 2.4.16 has been verified in many cases by the work of Kisin and Emerton. However, Conjecture 2.4.16 remains wide open for higher dimensional representations.

The local version of Conjecture 2.4.16 is known to be false. More precisely, there exists a de Rham representation of Γ_K which does not arises as a subquotient of $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)(m)$ for any proper smooth variety X over K and integers n, m.

3. Crystalline representations

In this section we define and study the crystalline period ring and crystalline representations. Our primary reference for this section is Brinon and Conrad's notes [**BC**, §9].

3.1. The crystalline period ring B_{cris}

Throughout this section, we write $W(k)$ for the ring of Witt vectors over k, and K_0 for its fraction field. Recall that we have fixed an element $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\sharp = p$ and set $\xi = [p^\flat] - p \in A_{\mathrm{inf}}$.

Definition 3.1.1. We define the *integral crystalline period ring* by

$$A_{\mathrm{cris}} := \left\{ \sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!} \in B_{\mathrm{dR}}^+ : a_n \in A_{\mathrm{inf}} \text{ with } \lim_{n\to\infty} a_n = 0 \right\},$$

and write $B_{\mathrm{cris}}^+ := A_{\mathrm{cris}}[1/p]$.

Remark. In the definition of A_{cris} above, it is vital to consider the refinement of the discrete valuation topology on B_{dR}^+ as described in Proposition 2.2.19. While the convergence of the infinite sum $\sum_{n\geq 0} a_n \frac{\xi^n}{n!}$ relies on the discrete valuation topology on B_{dR}^+, the limit of the coefficients a_n should be taken with respect to the p-adic topology on A_{inf}.

We warn the readers that the terminology given in Definition 3.1.1 is not standard at all. In fact, most authors do not give a separate name for the ring A_{cris}. Our choice of the terminology comes from the fact that A_{cris} plays the role of the crystalline period ring in the integral p-adic Hodge theory.

Proposition 3.1.2. We have $t \in A_{\mathrm{cris}}$ and $t^{p-1} \in pA_{\mathrm{cris}}$.

Proof. By Lemma 2.2.21 we may write $[\varepsilon] - 1 = \xi c$ for some $c \in A_{\mathrm{inf}}$. Then we obtain an expression

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} (n-1)! c^n \cdot \frac{\xi^n}{n!}. \tag{3.1}$$

We thus find $t \in A_{\mathrm{cris}}$ as we have $\lim_{n\to\infty} (n-1)! c^n = 0$ in A_{inf} relative to the p-adic topology.

It remains to show $t^{p-1} \in pA_{\mathrm{cris}}$. Let us set

$$\check{t} := \sum_{n=1}^{p} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n}. \tag{3.2}$$

Since $(n-1)!$ is divisible by p for all $n > p$, we find $t - \check{t} \in pA_{\mathrm{cris}}$ by (3.1). Hence it suffices to prove $\check{t}^{p-1} \in pA_{\mathrm{cris}}$.

The terms for $n < p$ in (3.2) are all divisible by $[\varepsilon]-1$ in A_{cris}, whereas the term for $n = p$ in (3.2) can be written as

$$(-1)^{p+1} \frac{([\varepsilon]-1)^p}{p} = (-1)^{p+1} \frac{([\varepsilon]-1)^{p-1}}{p} \cdot ([\varepsilon]-1).$$

In other words, we may write

$$\check{t} = ([\varepsilon]-1) \left(a + (-1)^{p+1} \frac{([\varepsilon]-1)^{p-1}}{p} \right)$$

for some $a \in A_{\mathrm{cris}}$. It is therefore enough to show $([\varepsilon]-1)^{p-1} \in pA_{\mathrm{cris}}$.

Since we have $([\varepsilon]-1)-[\varepsilon-1] \in pA_{\text{inf}} \subseteq pA_{\text{cris}}$, we only need to prove $[(\varepsilon-1)^{p-1}] \in pA_{\text{cris}}$. In addition, by Lemma 2.2.21 we have

$$\nu^\flat\left((\varepsilon-1)^{p-1}\right) = p = \nu^\flat\left((p^\flat)^p\right),$$

and consequently find that $[(\varepsilon-1)^{p-1}]$ is divisible by $[p^\flat]^p = (\xi+p)^p$. We thus deduce the desired assertion by observing that $\xi^p = p \cdot (p-1)! \cdot (\xi^p/p!)$ is divisible by p in A_{cris}. □

Remark. As a consequence, we find

$$\frac{t^p}{p!} = \frac{t^{p-1}}{p} \cdot \frac{t}{(p-1)!} \in A_{\text{cris}}.$$

In fact, it is not hard to prove that for every $a \in A_{\text{cris}}$ with $\theta_{\text{dR}}^+(a) = 0$ we have $a^n/n! \in A_{\text{cris}}$ for all $n \geq 1$.

Corollary 3.1.3. We have an identification $B_{\text{cris}}^+[1/t] = A_{\text{cris}}[1/t]$.

Proof. Proposition 3.1.2 implies that p is a unit in $A_{\text{cris}}[1/t]$, thereby yielding

$$B_{\text{cris}}^+[1/t] = A_{\text{cris}}[1/p, 1/t] = A_{\text{cris}}[1/t]$$

as desired. □

Definition 3.1.4. We define the *crystalline period ring* by

$$B_{\text{cris}} := B_{\text{cris}}^+[1/t] = A_{\text{cris}}[1/t].$$

Remark. Let us briefly explain Fontaine's insight behind the construction of B_{cris}. The main motivation for constructing the crystalline period ring B_{cris} is to obtain the Grothendieck mysterious functor as described in Chapter I. Recall that, for a proper smooth variety X over K with a proper smooth integral model $\mathcal{X}$ over $\mathcal{O}_K$, the crystalline cohomology $H^n_{\text{cris}}(\mathcal{X}_k, W(k))$ admits a natural Frobenius action and refines the de Rham cohomology $H^n_{\text{dR}}(X/K)$ via a canonical isomorphism

$$H^n_{\text{cris}}(\mathcal{X}_k, W(k))[1/p] \otimes_{K_0} K \cong H^n_{\text{dR}}(X/K).$$

In addition, since A_{inf} is by construction the ring of Witt vectors over a perfect $\mathbb{F}_p$-algebra $\mathcal{O}_F$, it admits the Frobenius automorphism φ_{inf} as noted in Chapter II, Proposition 2.3.4. Fontaine sought to construct B_{cris} as a sufficiently large subring of B_{dR} on which φ_{inf} naturally extends. For B_{dR} there is no natural extension of φ_{inf} since $\ker(\theta[1/p])$ is not stable under φ_{inf}. Fontaine's key observation is that by adjoining to A_{inf} the elements of the form $\xi^n/n!$ for $n \geq 1$ we obtain a subring of $A_{\text{inf}}[1/p]$ such that the image of $\ker(\theta[1/p])$ is stable under φ_{inf}. This observation led Fontaine to consider the ring A_{cris} defined in Definition 3.1.1. The only issue with A_{cris} is that it is not $(\mathbb{Q}_p, \Gamma_K)$-regular, which turns out to be resolved by considering the ring $B_{\text{cris}} = A_{\text{cris}}[1/t]$.

Proposition 3.1.5. The ring B_{cris} is naturally a filtered subalgebra of B_{dR} over K_0 which is stable under the action of Γ_K.

Proof. By construction we have

$$A_{\text{inf}}[1/p] \subseteq A_{\text{cris}}[1/p] = B_{\text{cris}}^+ \subseteq B_{\text{cris}} \subseteq B_{\text{dR}}.$$

In addition, the proof of Proposition 2.2.18 yields a unique homomorphism $K \longrightarrow B_{\text{dR}}$ which extends a natural homomorphism $K_0 \longrightarrow A_{\text{inf}}[1/p]$. Hence by Example 2.3.2 we naturally identify B_{cris} as a filtered subalgebra of B_{dR} over K_0 with $\text{Fil}^n(B_{\text{cris}}) := B_{\text{cris}} \cap t^n B_{\text{dR}}^+$.

It remains to show that $B_{\text{cris}} = A_{\text{cris}}[1/t]$ is stable under the action of Γ_K. Since Γ_K acts on t by the cyclotomic character as noted in Theorem 2.2.24, we only need to show that A_{cris}

is stable under the action of Γ_K. Consider an arbitrary element $\gamma \in \Gamma_K$ and an arbitrary sequence (a_n) in A_{inf} with $\lim\limits_{n\to\infty} a_n = 0$. Since $\ker(\theta)$ is stable under the Γ_K-action as noted in Theorem 2.2.24, we may write $\gamma(\xi) = c_\gamma \xi$ for some $c_\gamma \in A_{\mathrm{inf}}$ by Proposition 2.2.11. We then have $\lim\limits_{n\to\infty} \gamma(a_n)c_\gamma^n = 0$ as the Γ_K-action on A_{inf} is evidently continuous with respect to the p-adic topology. Hence we find

$$\gamma\left(\sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!}\right) = \sum_{n=0}^{\infty} \gamma(a_n) c_\gamma^n \frac{\xi^n}{n!} \in A_{\mathrm{cris}}$$

as desired. □

Remark. We provide a functorial perspective for the Γ_K-actions on B_{cris} and B_{dR} which can be useful in many occasions. Since the definitions of B_{cris} and B_{dR} only depend on the valued field $\mathbb{C}_K$, we may regard B_{cris} and B_{dR} as functors which associate topological rings to each complete and algebraically closed valued field. Then by functoriality the action of Γ_K on $\mathbb{C}_K$ induces the actions of Γ_K on B_{cris} and B_{dR}. In particular, since B_{cris} is a subfunctor of B_{dR} we deduce that the Γ_K-action on B_{cris} is given by the restriction of the Γ_K-action on B_{dR} as asserted in Proposition 3.1.5.

We also warn that $\mathrm{Fil}^0(B_{\mathrm{cris}}) = B_{\mathrm{cris}} \cap B_{\mathrm{dR}}^+$ is not equal to B_{cris}^+. For example, the element

$$\alpha = \frac{[\varepsilon^{1/p^2}] - 1}{[\varepsilon^{1/p}] - 1}$$

lies in $B_{\mathrm{cris}} \cap B_{\mathrm{dR}}^+$ but not in B_{cris}^+.

In order to study the Γ_K-action on B_{cris} we invoke the following crucial (and surprisingly technical) result without proof.

Proposition 3.1.6. The natural Γ_K-equivariant map $B_{\mathrm{cris}} \otimes_{K_0} K \longrightarrow B_{\mathrm{dR}}$ is injective.

Remark. The original proof by Fontaine in [**Fon94**] is incorrect. A complete proof involving the semistable period ring can be found in Fontaine and Ouyang's notes [**FO**, Theorem 6.14]. Note however that the assertion is obvious if we have $K = K_0$, which amounts to the condition that K is unramified over $\mathbb{Q}_p$.

Proposition 3.1.7. There exists a natural isomorphism of graded K-algebras

$$\mathrm{gr}(B_{\mathrm{cris}} \otimes_{K_0} K) \cong \mathrm{gr}(B_{\mathrm{dR}}) \cong B_{\mathrm{HT}}.$$

Proof. We only need to establish the first identification as the second identification immediately follows from Theorem 2.2.24 as noted in Example 2.3.2. By Proposition 3.1.6 the natural map $B_{\mathrm{cris}} \otimes_{K_0} K \longrightarrow B_{\mathrm{dR}}$ induces an injective morphism of graded K-algebras

$$\mathrm{gr}(B_{\mathrm{cris}} \otimes_{K_0} K) \hookrightarrow \mathrm{gr}(B_{\mathrm{dR}}). \tag{3.3}$$

In particular, we have an injective map

$$\mathrm{gr}^0(B_{\mathrm{cris}} \otimes_{K_0} K) \hookrightarrow \mathrm{gr}^0(B_{\mathrm{dR}}) \cong \mathbb{C}_K$$

where the isomorphism is induced by θ_{dR}^+. Moreover, this map is surjective since the image of $B_{\mathrm{cris}} \otimes_{K_0} K$ in B_{dR} contains $A_{\mathrm{inf}}[1/p]$ and consequently maps onto $\mathbb{C}_K$ by θ_{dR}^+. Therefore we obtain an isomorphism

$$\mathrm{gr}^0(B_{\mathrm{cris}} \otimes_{K_0} K) \cong \mathrm{gr}^0(B_{\mathrm{dR}}) \cong \mathbb{C}_K.$$

This implies that each $\mathrm{gr}^n(B_{\mathrm{cris}} \otimes_{K_0} K)$ is a vector space over $\mathbb{C}_K$. Moreover, each $\mathrm{gr}^n(B_{\mathrm{cris}} \otimes_{K_0} K)$ contains a nonzero element given by $t^n \otimes 1$. Hence the injective map (3.3) must be an isomorphism since each $\mathrm{gr}^n(B_{\mathrm{dR}})$ has dimension 1 over $\mathbb{C}_K$. □

Theorem 3.1.8 (Fontaine [**Fon94**]). The ring B_{cris} is $(\mathbb{Q}_p, \Gamma_K)$-regular with $B_{\text{cris}}^{\Gamma_K} \cong K_0$.

Proof. Let C_{cris} denote the fraction field of B_{cris}. Proposition 3.1.5 implies that C_{cris} is a subfield of B_{dR} which is stable under the action of Γ_K. Hence we have $K_0 \subseteq B_{\text{cris}}^{\Gamma_K} \subseteq C_{\text{cris}}^{\Gamma_K}$. Then Proposition 3.1.6 and Theorem 2.2.24 together yield injective maps

$$B_{\text{cris}}^{\Gamma_K} \otimes_{K_0} K \hookrightarrow B_{\text{dR}}^{\Gamma_K} \cong K \qquad \text{and} \qquad C_{\text{cris}}^{\Gamma_K} \otimes_{K_0} K \hookrightarrow B_{\text{dR}}^{\Gamma_K} \cong K,$$

thereby implying $K_0 = B_{\text{cris}}^{\Gamma_K} = C_{\text{cris}}^{\Gamma_K}$.

It remains to check the condition (ii) in Definition 1.1.1. Consider an arbitrary nonzero element $b \in B_{\text{cris}}$ on which Γ_K acts via a character $\eta : \Gamma_K \longrightarrow \mathbb{Q}_p^\times$. We wish to show that b is a unit in B_{cris}.

By Proposition 2.2.22 we may write $b = t^i b'$ for some $b' \in (B_{\text{dR}}^+)^\times$ and $i \in \mathbb{Z}$. Since t is a unit in B_{cris} by construction, the element b is a unit in B_{cris} if and only if b' is a unit in B_{cris}. Moreover, Theorem 2.2.24 implies that Γ_K acts on $b' = b \cdot t^{-i}$ via the character $\eta\chi^{-i}$. Hence we may replace b by b' to assume that b is a unit in B_{dR}^+.

Since θ_{dR}^+ is Γ_K-equivariant as noted in Theorem 2.2.24, the action of Γ_K on $\theta_{\text{dR}}^+(b) \in \mathbb{C}_K$ is given by the character η. Then by the continuity of the Γ_K-action on $\mathbb{C}_K$ we find that η is continuous. Therefore we may consider η as a character with values in $\mathbb{Z}_p^\times$. Moreover, we have $\theta_{\text{dR}}^+(b) \neq 0$ as b is assumed to be a unit in B_{dR}^+. Hence Theorem 1.1.8 implies that $\eta^{-1}(I_K)$ is finite.

Let us denote by K^{un} the maximal unramified extension of K in $\overline{K}$, and by $\widehat{K^{\text{un}}}$ the p-adic completion of K^{un}. By definition $\widehat{K^{\text{un}}}$ is a p-adic field with I_K as the absolute Galois group. Therefore by our discussion in the preceding paragraph there exists a finite extension L of $\widehat{K^{\text{un}}}$ with the absolute Galois group Γ_L such that η^{-1} becomes trivial on $\Gamma_L \subseteq I_K$. Since Γ_K acts on $\theta_{\text{dR}}^+(b)$ via η, we find $\theta_{\text{dR}}^+(b) \in \mathbb{C}_K^{\Gamma_L} = \mathbb{C}_L^{\Gamma_L} = L$ by Theorem 3.1.14 in Chapter II.

Let us write $W(\overline{k})$ for the ring of Witt vectors over $\overline{k}$, and $\widehat{K_0^{\text{un}}}$ for the fraction field of $W(\overline{k})$. Proposition 2.2.18 yields a commutative diagram

$$\begin{array}{ccc} \widehat{K_0^{\text{un}}} & \longrightarrow & A_{\text{inf}}[1/p] \\ \big\downarrow & & \big\downarrow \\ L & \longrightarrow & B_{\text{dR}}^+ \\ & \searrow & \big\downarrow \theta_{\text{dR}}^+ \\ & & \mathbb{C}_K \end{array} \tag{3.4}$$

where all maps are Γ_K-equivariant. Moreover, both horizontal maps are injective as $\widehat{K_0^{\text{un}}}$ and L are fields. We henceforth identify $\widehat{K_0^{\text{un}}}$ and L with their images in B_{dR}. Then we have

$$\widehat{K_0^{\text{un}}} \subseteq A_{\text{inf}}[1/p] \subseteq B_{\text{cris}}. \tag{3.5}$$

We assert that b lies in (the image of) L. Let us write $\hat{b} := \theta_{\text{dR}}^+(b)$. If suffices to show $b = \hat{b}$. Suppose for contradiction that b and $\hat{b}$ are distinct. Since we have $\theta_{\text{dR}}^+(\hat{b}) = \hat{b} = \theta_{\text{dR}}^+(b)$ by the commutativity of the diagram (3.4), we may write $b - \hat{b} = t^j u$ for some $j > 0$ and $u \in (B_{\text{dR}}^+)^\times$. Moreover, we find

$$\gamma(b - \hat{b}) = \gamma(b) - \gamma(\hat{b}) = \eta(\gamma)(b - \hat{b}) \qquad \text{for every } \gamma \in \Gamma_K.$$

Then under the Γ_K-equivariant isomorphism

$$t^j B_{\mathrm{dR}}^+/t^{j+1}B_{\mathrm{dR}}^+ \cong \mathbb{C}_K(j)$$

given by Theorem 2.2.24, the element $b - \hat{b} \in t^j B_{\mathrm{dR}}^+$ yields a nonzero element in $\mathbb{C}_K(j)$ on which Γ_K acts via the character η. Therefore Theorem 1.1.8 implies that $(\chi^j\eta^{-1})(I_K)$ is finite. Since $\eta^{-1}(I_K)$ is also finite as noted above, we deduce that $\chi^j(I_K)$ is finite as well, thereby obtaining a desired contradiction by Lemma 1.1.7.

Let us now regard b as an element in L. Proposition 2.2.18 implies that L is a finite extension of $\widehat{K_0^{\mathrm{un}}}$. Hence we can choose a minimal polynomial equation

$$b^d + a_1 b^{d-1} + \cdots + a_{d-1}b + a_d = 0 \qquad \text{with } a_n \in \widehat{K_0^{\mathrm{un}}}.$$

Since the minimality of the equation implies $a_d \neq 0$, we obtain an expression

$$b^{-1} = -a_d^{-1}(b^{d-1} + a_1 b^{d-2} + \cdots + a_{d-1}).$$

We then find $b^{-1} \in B_{\mathrm{cris}}$ by (3.5), thereby completing the proof. □

Our final goal in this subsection is to construct the Frobenius endomorphism on B_{cris}. To this end we state another technical result without proof.

Proposition 3.1.9. Let A_{cris}^0 be the A_{inf}-subalgebra in $A_{\mathrm{inf}}[1/p]$ generated by the elements of the form $\xi^n/n!$ with $n \geq 0$.

(1) The ring A_{cris} is naturally identified with the p-adic completion of A_{cris}^0.
(2) The action of Γ_K on A_{cris} is continuous.

Remark. In fact, Fontaine originally defined the ring A_{cris} as the p-adic completion of A_{cris}^0, and obtained an identification with our definition of A_{cris}. The proof requires yet another description of the ring A_{cris} as a p-adically completed tensor product. The readers can find a sketch of the proof in [**BC**, Proposition 9.1.1 and Proposition 9.1.2].

Lemma 3.1.10. The Frobenius automorphism of A_{inf} uniquely extends to a Γ_K-equivariant continuous endomorphism φ^+ on B_{cris}^+.

Proof. The Frobenius automorphism of A_{inf} uniquely extends to an automorphism on $A_{\mathrm{inf}}[1/p]$, which we denote by φ_{inf}. By construction we have

$$\varphi_{\mathrm{inf}}(\xi) = [(p^\flat)^p] - p = [p^\flat]^p - p = (\xi + p)^p - p. \tag{3.6}$$

Hence we may write $\varphi_{\mathrm{inf}}(\xi) = \xi^p + pc$ for some $c \in A_{\mathrm{inf}}$.

Let us define A_{cris}^0 as in Proposition 3.1.9. Then we have

$$\varphi_{\mathrm{inf}}(\xi) = p \cdot (c + (p-1)! \cdot (\xi^p/p!)),$$

and consequently find

$$\varphi_{\mathrm{inf}}(\xi^n/n!) = (p^n/n!) \cdot (c + (p-1)! \cdot (\xi^p/p!))^n \in A_{\mathrm{cris}}^0 \qquad \text{for all } n \geq 1$$

by observing that $p^n/n!$ is an element of $\mathbb{Z}_p$. Hence A_{cris}^0 is stable under φ_{inf}. Moreover, by construction φ_{inf} is Γ_K-equivariant and continuous on $A_{\mathrm{inf}}[1/p]$ with respect to the p-adic topology. We thus deduce by Proposition 3.1.9 that the endomorphism φ_{inf} on A_{cris}^0 uniquely extends to a continuous Γ_K-equivariant endomorphism φ^+ on $B_{\mathrm{cris}}^+ = A_{\mathrm{cris}}[1/p]$. □

Remark. The identity (3.6) shows that $\varphi_{\mathrm{inf}}(\xi)$ is not divisible by ξ, which implies that $\ker(\theta)$ is not stable under φ_{inf}. Hence the endomorphism φ^+ on B_{cris}^+ (or the Frobenius endomorphism on B_{cris} that we are about to construct) is not compatible with the filtration on B_{dR}.

Proposition 3.1.11. The Frobenius automorphism of A_{inf} naturally extends to a Γ_K-equivariant endomorphism φ on B_{cris} with $\varphi(t) = pt$.

Proof. As noted in Lemma 3.1.10, the Frobenius automorphism of A_{inf} uniquely extends to a Γ_K-equivariant continuous endomorphism φ^+ on B_{cris}^+. In addition, the proof of Proposition 3.1.2 shows that the power series expression

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon] - 1)^n}{n}$$

converges with respect to the p-adic topology in A_{cris}. Hence we use Proposition 2.2.22 and the continuity of φ^+ on A_{cris} to find

$$\varphi^+(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\varphi([\varepsilon]) - 1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon^p] - 1)^n}{n} = \log(\varepsilon^p) = p \log(\varepsilon) = pt.$$

Since Γ_K acts on t via χ, it follows that φ^+ uniquely extends to a Γ_K-equivariant endomorphism φ on $B_{\text{cris}} = B_{\text{cris}}^+[1/t]$. □

Remark. The endomorphism φ is not continuous on B_{cris}, even though it is a unique extension of the continuous endomorphism φ^+ on B_{cris}^+. The issue is that, as pointed out by Colmez in [**Col98**], the natural topology on B_{cris}^+ induced by the p-adic topology on A_{cris} does not agree with the subspace topology inherited from B_{cris}.

Definition 3.1.12. We refer to the endomorphism φ in Proposition 3.1.11 as the *Frobenius endomorphism* of B_{cris}. We also write

$$B_e := \{ b \in B_{\text{cris}} : \varphi(b) = b \}$$

for the ring of Frobenius-invariant elements in B_{cris}.

Remark. In Chapter IV, we will use the Fargues-Fontaine curve to prove a surprising fact that B_e is a principal ideal domain.

We close this subsection by stating two fundamental results about φ without proof.

Theorem 3.1.13. The Frobenius endomorphism φ of B_{cris} is injective.

Theorem 3.1.14. The natural sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow B_{\text{dR}}/B_{\text{dR}}^+ \longrightarrow 0$$

is exact.

Remark. We will prove both Theorem 3.1.13 and Theorem 3.1.14 in Chapter IV using the Fargues-Fontaine curve. There will be no circular reasoning; the construction of the Fargues-Fontaine curve does not rely on anything that we haven't discussed so far in this section. The readers can also find a proof of Theorem 3.1.14 in [**FO**, Theorem 6.26]. We also remark that, as mentioned in [**BC**, Theorem 9.1.8], there was no published proof of Theorem 3.1.13 prior to the work of Fargues-Fontaine [**FF18**].

Definition 3.1.15. We refer to the exact sequence in Theorem 3.1.14 as the *fundamental exact sequence of p-adic Hodge theory.*

3.2. Properties of crystalline representations

Definition 3.2.1. We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is *crystalline* if it is B_{cris}-admissible. We write $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K) := \mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{cris}}}(\Gamma_K)$ for the category of crystalline p-adic Γ_K-representations. In addition, we write D_{cris} the functors $D_{B_{\mathrm{cris}}}$.

Example 3.2.2. We record some essential examples of crystalline representations.

(1) Every Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ is crystalline; indeed, the inequality

$$\dim_K D_{\mathrm{cris}}(\mathbb{Q}_p(n)) \leq \dim_{\mathbb{Q}_p} \mathbb{Q}_p(n) = 1$$

given by Theorem 1.2.1 is an equality, as $D_{\mathrm{cris}}(\mathbb{Q}_p(n)) = (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K}$ contains a nonzero element $1 \otimes t^{-n}$ by Theorem 2.2.24.

(2) For every proper smooth variety X over K with with a proper smooth integral model $\mathcal{X}$ over $\mathcal{O}_K$, the étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline by a theorem of Faltings as discussed in Chapter I, Theorem 1.2.4; moreover, there exists a canonical isomorphism

$$D_{\mathrm{cris}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H_{\mathrm{cris}}(\mathcal{X}_k/K_0) = H^n_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p]$$

where $H_{\mathrm{cris}}(\mathcal{X}_k/W(k))$ denotes the crystalline cohomology of $\mathcal{X}_k$.

(3) For every p-divisible group G over $\mathcal{O}_K$, the rational Tate module $V_p(G)$ is crystalline as proved by Fontaine; indeed, there exists a natural identification

$$D_{\mathrm{cris}}(V_p(G)) \cong \mathbb{D}(\overline{G})[1/p]$$

where $\mathbb{D}(\overline{G})$ denotes the Dieudonné module associated to $\overline{G} := G \times_{\mathcal{O}_K} k$ as described in Chapter II, Theorem 2.3.12.

We aim to promote D_{cris} to a functor that incorporates both the Frobenius endomorphism and the filtration on B_{cris}. Let us denote by σ the Frobenius automorphism of K_0. The readers may wish to review the definition and basic properties of isocrystals as discussed in Chapter II, Definition 2.3.15 and Lemma 2.3.16.

Definition 3.2.3. A *filtered isocrystal* over K is an isocrystal N over K_0 together with a collection of K-spaces $\{ \mathrm{Fil}^n(N_K) \}$ which yields a structure of a filtered vector space over K on $N_K := N \otimes_{K_0} K$. We denote by MF_K^{φ} the category of filtered isocrystals over K with the natural notions of morphisms, tensor products, and duals inherited from the corresponding notions for Fil_K and the category of isocrystals over K_0.

Remark. Many authors use an alternative terminology *filtered φ-modules.*

Example 3.2.4. Let X be a proper smooth variety over K with a proper smooth integral model $\mathcal{X}$ over $\mathcal{O}_K$. The crystalline cohomology $H_{\mathrm{cris}}(\mathcal{X}_k/K_0) = H^n_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p]$ is naturally a filtered isocrystal over K with the Frobenius automorphism $\varphi^*_{\mathcal{X}_k}$ induced by the relative Frobenius of $\mathcal{X}_K$ and the filtration on $H^n_{\mathrm{cris}}(\mathcal{X}_k/K_0) \otimes_{K_0} K$ given by the Hodge filtration on the de Rham cohomology $H^n_{\mathrm{dR}}(X/K)$ via the canonical comparison isomorphism

$$H^n_{\mathrm{cris}}(\mathcal{X}_k/K_0) \otimes_{K_0} K \cong H^n_{\mathrm{dR}}(X/K).$$

Lemma 3.2.5. The automorphism σ on K_0 extends to the endomorphism φ on B_{cris}.

Proof. By the proof of Proposition 2.2.18, the natural injective map $K_0 \hookrightarrow A_{\mathrm{inf}}[1/p]$ is a unique lift of the natural map $k \longrightarrow \mathcal{O}_F$. Hence σ extends to φ_{inf} on $A_{\mathrm{inf}}[1/p]$ by definition, and consequently extends to φ by Proposition 3.1.12. ☐

LEMMA 3.2.6. Let N be a finite dimensional vector space over K_0. Every injective σ-semilinear additive map $f : N \longrightarrow N$ is bijective.

PROOF. The additivity of f implies that $f(N)$ is closed under addition. Moreover, for all $c \in K_0$ and $n \in N$ we have

$$cf(n) = \sigma(\sigma^{-1}(c))f(n) = f(\sigma^{-1}(c)n) \in f(N).$$

Therefore $f(N)$ is a subspace of N over K_0. We wish to show $f(N) = N$. Let us choose a basis (n_i) for N over K_0. It suffices to prove that the vectors $f(n_i)$ are linearly independent over K_0. Assume for contradiction that there exists a nontrivial a relation $\sum c_i f(n_i) = 0$ with $c_i \in K_0$. Then we find $f\left(\sum \sigma(c_i)n_i\right) = 0$ by the σ-semilinearity of f, and consequently obtain a relation $\sum \sigma(c_i)n_i = 0$ by the injectivity of f. Hence we have a nontrivial relation among the vectors n_i as σ is an automorphism on K_0, thereby obtaining contradiction as desired. □

PROPOSITION 3.2.7. Let V be a p-adic representation of Γ_K. Then $D_{\text{cris}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_K}$ is naturally a filtered isocrystal over K with the Frobenius automorphism $1 \otimes \varphi$ and the filtration on $D_{\text{cris}}(V)_K = D_{\text{cris}}(V) \otimes_{K_0} K$ given by

$$\text{Fil}^n(D_{\text{cris}}(V)_K) := (V \otimes_{\mathbb{Q}_p} \text{Fil}^n(B_{\text{cris}} \otimes_{K_0} K))^{\Gamma_K}.$$

PROOF. Since Γ_K acts trivially on K, we have a natural identification

$$D_{\text{cris}}(V)_K = (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_K} \otimes_{K_0} K = (V \otimes_{\mathbb{Q}_p} (B_{\text{cris}} \otimes_{K_0} K))^{\Gamma_K}.$$

Then Proposition 3.1.6 implies that $D_{\text{cris}}(V)_K$ is a filtered vector space over K with the filtration $\text{Fil}^n(D_{\text{cris}}(V)_K)$ as defined above. Therefore it remains to verify that the map $1 \otimes \varphi$ is σ-semilinear and bijective on $D_{\text{cris}}(V)$. For arbitrary $v \in V, b \in B_{\text{cris}}$, and $c \in K_0$ we have

$$(1 \otimes \varphi)(c(v \otimes b)) = (1 \otimes \varphi)(v \otimes bc) = v \otimes \varphi(b)\varphi(c) = \varphi(c) \cdot (1 \otimes \varphi)(v \otimes b).$$

Hence by Lemma 3.2.5 we find that the additive map $1 \otimes \varphi$ is σ-semilinear. Moreover, the map $1 \otimes \varphi$ is injective on $D_{\text{cris}}(K)$ by Theorem 3.1.13 and the left exactness of the functor D_{cris}. Thus we deduce the desired assertion by Lemma 3.2.6. □

PROPOSITION 3.2.8. Let V be a crystalline representation of Γ_K. Then V is de Rham with a natural isomorphism of filtered vector spaces

$$D_{\text{cris}}(V)_K = D_{\text{cris}}(V) \otimes_{K_0} K \cong D_{\text{dR}}(V).$$

PROOF. Proposition 3.1.5 and Proposition 3.1.6 together imply that the natural map $B_{\text{cris}} \otimes_{K_0} K \longrightarrow B_{\text{dR}}$ identifies $B_{\text{cris}} \otimes_{K_0} K$ as a filtered subspace of B_{dR} over K; in other words, we have an identification

$$\text{Fil}^n(B_{\text{cris}} \otimes_{K_0} K) = (B_{\text{cris}} \otimes_{K_0} K) \cap \text{Fil}^n(B_{\text{dR}}) \qquad \text{for every } n \in \mathbb{Z}.$$

Therefore Proposition 3.2.7 yields a natural injective morphism of filtered vector spaces

$$D_{\text{cris}}(V)_K = (V \otimes_{\mathbb{Q}_p} (B_{\text{cris}} \otimes_{K_0} K))^{\Gamma_K} \hookrightarrow (V \otimes_{\mathbb{Q}_p} B_{\text{dR}})^{\Gamma_K} = D_{\text{dR}}(V)$$

with an identification

$$\text{Fil}^n(D_{\text{cris}}(V) \otimes_{K_0} K) = (D_{\text{cris}}(V) \otimes_{K_0} K) \cap \text{Fil}^n(D_{\text{dR}}(V)) \qquad \text{for every } n \in \mathbb{Z}.$$

We then find

$$\dim_{K_0} D_{\text{cris}}(V) = \dim_K D_{\text{cris}}(V)_K \leq \dim_K D_{\text{dR}}(V) \leq \dim_{\mathbb{Q}_p} V$$

where the last inequality follows from Theorem 1.2.1. Since V is crystalline, both inequalities should be in fact equalities, thereby yielding the desired assertion. □

Example 3.2.9. Let $\eta : \Gamma_K \longrightarrow \mathbb{Q}_p^\times$ be a nontrivial continuous character which factors through $\mathrm{Gal}(L/K)$ for some totally ramified finite extension L of K. Then $\mathbb{Q}_p(\eta)$ is de Rham by Proposition 2.4.14. We assert that $\mathbb{Q}_p(\eta)$ is not crystalline. Let us write Γ_L for the absolute Galois group of L. Since L is totally ramified over K, we have $B_{\mathrm{cris}}^{\Gamma_L} \cong K_0$ by Theorem 3.1.8 and the fact that the construction of B_{cris} depends only on $\mathbb{C}_K$. Moreover, we have $\mathbb{Q}_p(\eta)^{\Gamma_L} = \mathbb{Q}_p(\eta)$ and $\mathbb{Q}_p(\eta)^{\mathrm{Gal}(L/K)} = 0$ by construction. Hence we find an identification

$$\begin{aligned} D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)) &= (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K} = \left((\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_L}\right)^{\mathrm{Gal}(L/K)} \\ &= \left(\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}^{\Gamma_L}\right)^{\mathrm{Gal}(L/K)} \cong (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} K_0)^{\mathrm{Gal}(L/K)} \\ &= \mathbb{Q}_p(\eta)^{\mathrm{Gal}(L/K)} \otimes_{\mathbb{Q}_p} K_0 = 0, \end{aligned}$$

thereby deducing the desired assertion.

We now adapt the argument in §2.4 to verify that the general formalism discussed in §1 extends to the category of crystalline representations with the enhanced functor D_{cris} that takes values in MF_K^{φ}.

Proposition 3.2.10. Every $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ induces a natural Γ_K-equivariant isomorphism

$$D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$$

which is compatible with the natural Frobenius endomorphisms on both sides and induces a K-linear isomorphism of filtered vector spaces

$$D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K).$$

Proof. Since V is crystalline, Theorem 1.2.1 implies that the natural map

$$D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \longrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}) \otimes_{K_0} B_{\mathrm{cris}} \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} B_{\mathrm{cris}}) \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$$

is a Γ_K-equivariant B_{cris}-linear isomorphism. Moreover, this map is visibly compatible with the natural Frobenius endomorphisms on $D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K} \otimes_{K_0} B_{\mathrm{cris}}$ and $V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$ respectively given by $1 \otimes \varphi \otimes \varphi$ and $1 \otimes \varphi$. Let us now consider the induced K-linear bijective map

$$(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K) \longrightarrow V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K).$$

It is straightforward to check that this map is a morphism of filtered vector spaces. Therefore by Proposition 2.3.8 it suffices to show that the induced map

$$\mathrm{gr}\left(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)\right) \longrightarrow \mathrm{gr}\left(V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K)\right) \tag{3.7}$$

is an isomorphism. As V is crystalline, it is also Hodge-Tate with the natural isomorphism of graded vector spaces

$$\mathrm{gr}(D_{\mathrm{cris}}(V)_K) \cong \mathrm{gr}(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{HT}}(V)$$

by Proposition 3.2.8 and Proposition 2.4.4. Hence Proposition 2.3.10 and Proposition 3.1.7 together yield identifications

$$\begin{aligned} \mathrm{gr}\left(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)\right) &\cong \mathrm{gr}(D_{\mathrm{cris}}(V)_K) \otimes_K \mathrm{gr}(B_{\mathrm{cris}} \otimes_{K_0} K) \cong D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}}, \\ \mathrm{gr}\left(V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K)\right) &\cong V \otimes_{\mathbb{Q}_p} \mathrm{gr}(B_{\mathrm{cris}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}. \end{aligned}$$

We thus identify the map (3.7) with the natural map

$$D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}} \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}$$

given by Theorem 1.2.1, thereby deducing the desired assertion by the fact that V is Hodge-Tate. □

Proposition 3.2.11. The functor D_{cris} with values in MF_K^{φ} is faithful and exact on $\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$.

Proof. Let Vect_{K_0} denote the category of finite dimensional vector spaces over K_0. The faithfulness of D_{cris} on $\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$ follows immediately from Proposition 1.2.2 since the forgetful functor $\text{MF}_K^{\varphi} \longrightarrow \text{Vect}_{K_0}$ is faithful. Hence it remains to verify the exactness of D_{cris} on $\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$. Consider an arbitrary exact sequence of crystalline representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$$

We wish to show that the sequence

$$0 \longrightarrow D_{\text{cris}}(U) \longrightarrow D_{\text{cris}}(V) \longrightarrow D_{\text{cris}}(W) \longrightarrow 0 \tag{3.8}$$

is exact in MF_K^{φ}. This sequence is exact in Vect_{K_0} by Proposition 1.2.2, and thus is also exact in the category of isocrystals over K_0. Moreover, Proposition 3.2.8 and Proposition 2.4.9 together imply that we can identify the induced sequence of filtered vector spaces

$$0 \longrightarrow D_{\text{cris}}(U)_K \longrightarrow D_{\text{cris}}(V)_K \longrightarrow D_{\text{cris}}(W)_K \longrightarrow 0$$

with the exact sequence of filtered vector spaces

$$0 \longrightarrow D_{\text{dR}}(U) \longrightarrow D_{\text{dR}}(V) \longrightarrow D_{\text{dR}}(W) \longrightarrow 0$$

induced by (3.2). We thus deduce that the sequence (3.8) is exact in MF_K^{φ} as desired. □

Corollary 3.2.12. Let V be a crystalline representation. Every subquotient W of V is a crystalline representation with $D_{\text{cris}}(W)$ naturally identified as a subquotient of $D_{\text{dR}}(V)$.

Proof. This is an immediate consequence of Proposition 1.2.3 and Proposition 3.2.11. □

Proposition 3.2.13. Given any $V, W \in \text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$, we have $V \otimes_{\mathbb{Q}_p} W \in \text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$ with a natural isomorphism of filtered isocrystals

$$D_{\text{cris}}(V) \otimes_{K_0} D_{\text{cris}}(W) \cong D_{\text{cris}}(V \otimes_{\mathbb{Q}_p} W). \tag{3.9}$$

Proof. By Proposition 1.2.4 we find $V \otimes_{\mathbb{Q}_p} W \in \text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$ and obtain the desired isomorphism (3.9) as a map of vector spaces. Moreover, since the construction of the map (3.9) rests on the multiplicative structure of B_{cris} as shown in the proof of Proposition 1.2.4, it is straightforward to verify that the map (3.9) is a morphism of isocrystals over K_0. In addition, Proposition 3.2.8 implies that we can identify the induced bijective K-linear map

$$D_{\text{cris}}(V)_K \otimes_K D_{\text{cris}}(W)_K \longrightarrow D_{\text{cris}}(V \otimes_{\mathbb{Q}_p} W)_K.$$

with the natural isomorphism of filtered vector spaces

$$D_{\text{dR}}(V) \otimes_K D_{\text{dR}}(W)_K \cong D_{\text{dR}}(V \otimes_{\mathbb{Q}_p} W)$$

given by Proposition 2.4.11. Therefore we deduce that the map (3.9) is an isomorphism in MF_K^{φ} as desired. □

Proposition 3.2.14. For every crystalline representation V, we have $\wedge^n(V) \in \text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$ and $\text{Sym}^n(V) \in \text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$ with natural isomorphisms of filtered isocrystals

$$\wedge^n(D_{\text{cris}}(V)) \cong D_{\text{cris}}(\wedge^n(V)) \qquad \text{and} \qquad \text{Sym}^n(D_{\text{cris}}(V)) \cong D_{\text{cris}}(\text{Sym}^n(V)).$$

Proof. Proposition 1.2.5 implies that both $\wedge^n(V)$ and $\text{Sym}^n(V)$ are crystalline for every $n \geq 1$. In addition, Proposition 1.2.5 yields the desired isomorphisms as maps of vector spaces. Then Corollary 3.2.12 and Proposition 3.2.13 together imply that these maps are isomorphisms in MF_K^{φ}. □

PROPOSITION 3.2.15. For every crystalline representation V, the dual representation $V^\vee$ is crystalline with a natural perfect pairing of filtered isocrystals

$$D_{\text{cris}}(V) \otimes_{K_0} D_{\text{cris}}(V^\vee) \cong D_{\text{cris}}(V \otimes_{\mathbb{Q}_p} V^\vee) \longrightarrow D_{\text{cris}}(\mathbb{Q}_p).$$

PROOF. By Proposition 1.2.6 we find $V^\vee \in \mathrm{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$ and obtain the desired perfect pairing as a map of vector spaces. Moreover, Proposition 3.2.13 implies that this pairing is a morphism in MF_K^φ. We thus obtain a bijective morphism of filtered isocrystals

$$D_{\text{cris}}(V)^\vee \longrightarrow D_{\text{cris}}(V^\vee). \tag{3.10}$$

Furthermore, by Proposition 3.2.8 we identify the induced morphism of filtered vector spaces

$$D_{\text{cris}}(V)_K^\vee \longrightarrow D_{\text{cris}}(V^\vee)_K$$

with the natural isomorphism $D_{\text{dR}}(V) \cong D_{\text{dR}}(V^\vee)$ in Fil_K given by Proposition 2.4.13. Hence we deduce that the map (3.10) is an isomorphism in MF_K^φ, thereby completing the proof. □

Finally, we discuss some additional key properties of crystalline representations and the functor D_{cris} which resolve the main defects of de Rham representations and the functor D_{dR}.

Definition 3.2.16. Let M be a module over a ring R with an additive endomorphism f. For every $r \in R$, we refer to the subgroup

$$M^{f=r} := \{\, m \in M : f(m) = rm \,\}$$

as the *eigenspace of* f with eigenvalue r.

LEMMA 3.2.17. We have an identification

$$B_{\text{cris}}^{\varphi=1} \cap \mathrm{Fil}^0(B_{\text{cris}} \otimes_{K_0} K) = B_{\text{cris}}^{\varphi=1} \cap B_{\text{dR}}^+ = \mathbb{Q}_p.$$

PROOF. By Proposition 3.1.6 and Theorem 3.1.14 we find

$$B_{\text{cris}}^{\varphi=1} \cap \mathrm{Fil}^0(B_{\text{cris}} \otimes_{K_0} K) \subseteq B_{\text{cris}}^{\varphi=1} \cap \mathrm{Fil}^0(B_{\text{dR}}) = B_{\text{cris}}^{\varphi=1} \cap B_{\text{dR}}^+ = \mathbb{Q}_p,$$

and thus obtain the desired identification as both $B_{\text{cris}}^{\varphi=1}$ and $\mathrm{Fil}^0(B_{\text{cris}} \otimes_{K_0} K)$ contain $\mathbb{Q}_p$. □

PROPOSITION 3.2.18. Every $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$ admits canonical isomorphisms

$$\begin{aligned} V &\cong (D_{\text{cris}}(V) \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \cap \mathrm{Fil}^0\left(D_{\text{cris}}(V)_K \otimes_K (B_{\text{cris}} \otimes_{K_0} K)\right) \\ &\cong (D_{\text{cris}}(V) \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \cap \mathrm{Fil}^0\left(D_{\text{cris}}(V)_K \otimes_K B_{\text{dR}}\right). \end{aligned}$$

PROOF. Proposition 3.2.10 yields a natural Γ_K-equivariant isomorphism

$$D_{\text{cris}}(V) \otimes_{K_0} B_{\text{cris}} \cong V \otimes_{\mathbb{Q}_p} B_{\text{cris}}$$

which is compatible with the natural Frobenius endomorphisms on both sides and induces an isomorphism of filtered vector spaces

$$D_{\text{cris}}(V)_K \otimes_K (B_{\text{cris}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} (B_{\text{cris}} \otimes_{K_0} K).$$

In addition, there exists a canonical isomorphism of filtered vector spaces

$$D_{\text{cris}}(V)_K \otimes_K B_{\text{dR}} \cong D_{\text{dR}}(V) \otimes_K B_{\text{dR}} \cong V \otimes_{\mathbb{Q}_p} B_{\text{dR}}$$

given by Proposition 3.2.8 and Proposition 2.4.8. Therefore we have identifications

$$\begin{aligned} (D_{\text{cris}}(V) \otimes_{K_0} B_{\text{cris}})^{\varphi=1} &\cong V \otimes_{\mathbb{Q}_p} B_{\text{cris}}^{\varphi=1}, \\ \mathrm{Fil}^0\left(D_{\text{cris}}(V)_K \otimes_K (B_{\text{cris}} \otimes_{K_0} K)\right) &\cong V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^0(B_{\text{cris}} \otimes_{K_0} K), \\ \mathrm{Fil}^0\left(D_{\text{cris}}(V)_K \otimes_K B_{\text{dR}}\right) &\cong V \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+. \end{aligned}$$

The desired assertion now follows by Lemma 3.2.17. □

THEOREM 3.2.19 (Fontaine [**Fon94**]). The functor D_{cris} with values in MF_K^{φ} is exact and fully faithful on $\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$.

PROOF. By Proposition 3.2.11 we only need to establish the fullness of D_{cris} on $\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$. Let V and W be arbitrary crystalline representations. Consider an arbitrary morphism $f : D_{\text{cris}}(V) \longrightarrow D_{\text{cris}}(W)$ in MF_K^{φ}. Then f gives rise to a Γ_K-equivariant map

$$V \otimes_{\mathbb{Q}_p} B_{\text{cris}} \cong D_{\text{cris}}(V) \otimes_{K_0} B_{\text{cris}} \xrightarrow{f \otimes 1} D_{\text{cris}}(W) \otimes_{K_0} B_{\text{cris}} \cong W \otimes_{\mathbb{Q}_p} B_{\text{cris}} \tag{3.11}$$

where the isomorphisms are given by Proposition 3.2.10. Moreover, Proposition 3.2.18 implies that this map restricts to a linear map $\phi : V \longrightarrow W$. In other words, we may identify the map (3.11) as $\phi \otimes 1$. In particular, since the isomorphisms in (3.11) are Γ_K-equivariant, we recover f as the restriction of $\phi \otimes 1$ on $(V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_K} \cong (D_{\text{cris}}(V) \otimes_{K_0} B_{\text{cris}})^{\Gamma_K} \cong D_{\text{cris}}(V)$. This precisely means that f is induced by ϕ via the functor D_{cris}. □

PROPOSITION 3.2.20. Let V be a p-adic representation of Γ_K. Let L be a finite unramified extension of K with the residue field extension l of k. Denote by Γ_L the absolute Galois group of L and by L_0 the fraction field of the ring of Witt vectors over l.

(1) There exists a natural isomorphism of filtered isocrystals

$$D_{\text{cris},K}(V) \otimes_{K_0} L_0 \cong D_{\text{cris},L}(V)$$

where we set $D_{\text{cris},K}(V) := (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_K}$ and $D_{\text{cris},L}(V) := (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_L}$.

(2) V is crystalline if and only if it is crystalline as a representation of Γ_L.

PROOF. We only need to prove the first statement, as the second statement immediately follows from the first statement. By definition L and L_0 are respectively unramified extensions of K and K_0 with the residue field extension l of k. Hence L and L_0 are respectively Galois over K and K_0 with $\text{Gal}(L/K) \cong \text{Gal}(L_0/K_0)$. Furthermore, since the construction of B_{cris} depends only on $\mathbb{C}_K$, we have an identification

$$D_{\text{cris},K}(V) = D_{\text{cris},L}(V)^{\text{Gal}(L/K)} = D_{\text{cris},L}(V)^{\text{Gal}(L_0/K_0)}.$$

Then by the Galois descent for vector spaces we obtain a natural bijective L_0-linear map

$$D_{\text{cris},K}(V) \otimes_{K_0} L_0 \longrightarrow D_{\text{cris},L}(V). \tag{3.12}$$

This map is evidently compatible with the natural Frobenius automorphisms on both sides induced by φ as explained in Lemma 3.2.5 and Proposition 3.2.7. Moreover, Proposition 2.4.14 and Proposition 3.2.8 together imply that the map (3.12) induces a natural L-linear isomorphism of filtered vector spaces

$$(D_{\text{cris},K}(V) \otimes_{K_0} K) \otimes_K L \cong D_{\text{cris},L}(V) \otimes_{L_0} L.$$

We thus deduce that the map (3.12) is an isomorphism of filtered isocrystals over L. □

Remark. Proposition 3.2.20 also holds when L is the completion of the maximal unramified extension of K. As a consequence, we have the following facts:

(1) Every unramified p-adic representation is crystalline.

(2) For a continuous character $\eta : \Gamma_K \longrightarrow \mathbb{Z}_p^{\times}$, we have $\mathbb{Q}_p(\eta) \in \text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$ if and only if there exists some $n \in \mathbb{Z}$ such that $\eta\chi^n$ is trivial on I_K.

On the other hand, Example 3.2.9 shows that Proposition 3.2.20 fails when L is a ramified extension of K. Fontaine interpreted this "failure" as a good feature of the crystalline condition, and conjectured that the crystalline condition should provide a p-adic analogue of the Néron-Ogg-Shafarevich criterion introduced in Theorem 1.1.2 of Chapter I; more precisely, Fontaine conjectured that an abelian variety A over K has good reduction if and only if the rational Tate module $V_p(A[p^\infty])$ is crystalline. Fontaine's conjecture is now known to be true by the work of Coleman-Iovita and Breuil.

We conclude this section with a discussion of a classical example which is enlightening in many ways. We assume the following technical result without proof.

PROPOSITION 3.2.21. The continuous map $\log : \mathbb{Z}_p(1) \longrightarrow B_{\mathrm{dR}}^+$ extends to a Γ_K-equivariant homomorphism $\log : A_{\mathrm{inf}}[1/p]^\times \longrightarrow B_{\mathrm{dR}}^+$ such that $\log([p^\flat])$ is transcendental over the fraction field of B_{cris}.

Example 3.2.22. The *Tate curve* E_p is an elliptic curve over K with $E_p(\overline{K}) \cong \overline{K}^\times/p^{\mathbb{Z}}$ where we set $p^{\mathbb{Z}} := \{\, p^n : n \in \mathbb{Z} \,\}$. We assert that the rational Tate module $V_p(E_p[p^\infty])$ is de Rham but not crystalline. It is evident by construction that ε and $p^\flat$ form a basis of $V_p(E_p[p^\infty])$ over $\mathbb{Q}_p$. Moreover, for every $\gamma \in \Gamma_K$ we have

$$\gamma(\varepsilon) = \varepsilon^{\chi(\gamma)} \qquad \text{and} \qquad \gamma(p^\flat) = p^\flat \varepsilon^{c(\gamma)} \tag{3.13}$$

for some $c(\gamma) \in \mathbb{Z}_p$. Hence $V_p(E_p[p^\infty])$ is an extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ in $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, and thus is de Rham by Example 2.4.5.

We aim to present a basis for $D_{\mathrm{dR}}(V_p(E_p[p^\infty])) = (V_p(E_p[p^\infty]) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K}$. By (3.13) we find $\varepsilon \otimes t^{-1} \in D_{\mathrm{dR}}(V_p(E_p[p^\infty]))$. Let us now consider the homomorphism $\log : A_{\mathrm{inf}}[1/p]^\times \longrightarrow B_{\mathrm{dR}}^+$ as in Proposition 3.2.21 and set $u := \log([p^\flat])$. Then for $\gamma \in \Gamma_K$ we find

$$\gamma(u) = \gamma(\log([p^\flat]) = \log([\gamma(p^\flat)]) = \log([p^\flat \varepsilon^{c(\gamma)}]) = \log([p^\flat]) + c(\gamma)\log([\varepsilon]) = u + c(\gamma)t$$

by (3.13) and Proposition 2.2.22, and consequently obtain

$$\begin{aligned}\gamma(-\varepsilon \otimes ut^{-1} + p^\flat \otimes 1) &= -\varepsilon^{\chi(\gamma)} \otimes (u + c(\gamma)t)\chi(\gamma)^{-1}t^{-1} + p^\flat \varepsilon^{c(\gamma)} \otimes 1 \\ &= -\varepsilon \otimes (ut^{-1} + c(\gamma)) + c(\gamma) \cdot (\varepsilon \otimes 1) + p^\flat \otimes 1 \\ &= -\varepsilon \otimes ut^{-1} + p^\flat \otimes 1\end{aligned}$$

by (3.13) and Theorem 2.2.24. In particular, we have $-\varepsilon \otimes ut^{-1} + p^\flat \otimes 1 \in D_{\mathrm{dR}}(V_p(E_p[p^\infty]))$. Since the elements $\varepsilon \otimes t^{-1}$ and $-\varepsilon \otimes ut^{-1} + p^\flat \otimes 1$ are linearly independent over B_{dR}, they form a basis for $D_{\mathrm{dR}}(V_p(E_p[p^\infty])) = (V_p(E_p[p^\infty]) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K}$.

Let us now consider an arbitrary element $x \in D_{\mathrm{cris}}(V_p(E_p[p^\infty])) = (V_p(E_p[p^\infty]) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K}$. We may uniquely write $x = \varepsilon \otimes c + p^\flat \otimes d$ for some $c, d \in B_{\mathrm{cris}}$. Moreover, since we have $D_{\mathrm{cris}}(V_p(E_p[p^\infty])) \subseteq D_{\mathrm{dR}}(V_p(E_p[p^\infty]))$ there exist some $r, s \in K$ with

$$x = r \cdot (\varepsilon \otimes t^{-1}) + s \cdot (-\varepsilon \otimes ut^{-1} + p^\flat \otimes 1) = \varepsilon \otimes (r - su)t^{-1} + p^\flat \otimes s.$$

Then we find $c = (r - su)t^{-1}$, and consequently obtain $s = 0$ by Proposition 3.2.21. Therefore we deduce that every element in $D_{\mathrm{cris}}(V_p(E_p[p^\infty])) \otimes_{K_0} K$ is a K-multiple of $\varepsilon \otimes t^{-1}$. In particular, we find $\dim_{K_0} D_{\mathrm{cris}}(V_p(E_p[p^\infty])) \leq 1$, thereby concluding that $V_p(E_p[p^\infty])$ is not crystalline.

Remark. Fontaine constructed the *semistable period ring* B_{st} as the B_{cris}-subalgebra of B_{dR} generated by $\log([p^\flat])$.

CHAPTER IV

The Fargues-Fontaine curve

1. Construction

Our main objective in this section is to discuss the construction of the Fargues-Fontaine curve. The primary references are Fargues and Fontaine's survey paper [**FF12**] and Lurie's notes [**Lur**].

1.1. Untilts of a perfectoid field

Throughout this chapter, we let F be an algebraically closed perfectoid field F of characteristic p with the valuation ν_F, and write $\mathfrak{m}_F$ for the maximal ideal of $\mathcal{O}_F$. We also denote by $A_{\text{inf}} = W(\mathcal{O}_F)$ the ring of Witt vectors over $\mathcal{O}_F$, and by $W(F)$ the ring of Witt vectors over F. In addition, for every $c \in F$ we write $[c]$ for its Teichmüller lift in $W(F)$.

Definition 1.1.1. An *untilt* of F is a perfectoid field C together with a continuous isomorphism $\iota : F \simeq C^\flat$.

Example 1.1.2. The *trivial untilt* of F is the field F with the natural isomorphism $F \cong F^\flat$ given by Proposition 2.1.13 in Chapter III.

Definition 1.1.3. Let C be an untilt of F with a continuous isomorphism $\iota : F \simeq C^\flat$.

(1) We define the *sharp map* associated to C as the composition of the maps

$$F \xrightarrow[\iota]{\sim} C^\flat = \varprojlim_{x \mapsto x^p} C \longrightarrow C$$

where the last arrow is the projection to the first component.

(2) For every $c \in F$, we denote its image under the sharp map by $c^{\sharp_C}$, or often by $c^\sharp$.

(3) We define the *normalized valuation* on C to be the unique valuation ν_C with $\nu_F(c) = \nu_C(c^\sharp)$ for all $c \in F$ as given by Proposition 2.1.7 from Chapter III.

Our first goal in this subsection is to prove that every untilt of F is algebraically closed.

LEMMA 1.1.4. Let L be a complete nonarchimedean field, and let $f(x)$ be an irreducible monic polynomial over L with $f(0) \in \mathcal{O}_L$. Then $f(x)$ is a polynomial over $\mathcal{O}_L$.

PROOF. Let us choose a valuation ν_L on L. Take a finite Galois extension L' of L such that $f(x)$ factors as

$$f(x) = \prod_{i=1}^{d} (x - r_i) \qquad \text{with } r_i \in L'.$$

The valuation ν_L uniquely extends to a $\text{Gal}(L'/L)$-equivariant valuation $\nu_{L'}$ on L'. In particular, the roots r_i all have the same valuation as they belong to the same $\text{Gal}(L'/L)$-orbit. Since we have $f(0) = (-1)^d r_1 r_2 \cdots r_d \in \mathcal{O}_L$, we find that each r_i has a nonnegative valuation. Hence each coefficient of $f(x)$ has a nonnegative valuation as well. $\square$

Proposition 1.1.5. Let C be an untilt of F, and let $f(x)$ be an irreducible monic polynomial of degree d over C. For every $y \in C$, there exists an element $z \in C$ with

$$\nu_C(y - z) \geq \nu_C(f(y))/d \qquad \text{and} \qquad \nu_C(f(z)) \geq \nu_C(p) + \nu_C(f(y)).$$

Proof. We may replace $f(x)$ by $f(x + y)$ to assume $y = 0$. Our assertion is that there exists an element $z \in C$ with

$$\nu_C(z) \geq \nu_C(f(0))/d \qquad \text{and} \qquad \nu_C(f(z)) \geq \nu_C(p) + \nu_C(f(0)). \tag{1.1}$$

If we have $f(0) = 0$, the assertion is trivial as we can simply take $z = 0$. We henceforth assume $f(0) \neq 0$. Since F is algebraically closed, the multiplication by d is surjective on the value group of F. Hence Proposition 2.1.10 in Chapter III implies that the multiplication by d is also surjective on the value group of C. In particular, there exists an element $a \in C$ with $d\nu_C(a) = \nu_C(f(0))$. Then we can rewrite the inequalities in (1.1) as

$$\nu_C(z/a) \geq 0 \qquad \text{and} \qquad \nu_C\left(f(a \cdot (z/a))/a^d\right) \geq \nu_C(p).$$

Therefore we may replace $f(x)$ by the monic polynomial $f(a \cdot x)/a^d$ to assume $\nu_C(f(0)) = 0$. Then our assertion amounts to the existence of an element $z \in \mathcal{O}_C$ with $f(z) \in p\mathcal{O}_C$.

Lemma 1.1.4 implies that $f(x)$ is a polynomial over $\mathcal{O}_C$. In other words, we may write $f(x) = x^d + a_1 x^{d-1} + \cdots + a_d$ with $a_i \in \mathcal{O}_C$. Then by Lemma 2.1.8 in Chapter III we find elements $c_i \in \mathcal{O}_F$ with $a_i - c_i^\sharp \in p\mathcal{O}_C$. Since F is algebraically closed, the polynomial $g(x) := x^d + c_1 x^{d-1} + \cdots + c_d$ over $\mathcal{O}_F$ has a root α in $\mathcal{O}_F$. Now we find

$$\begin{aligned} f(\alpha^\sharp) &= (\alpha^\sharp)^d + a_1(\alpha^\sharp)^{d-1} + \cdots + a_d \\ &= (\alpha^\sharp)^d + c_1^\sharp(\alpha^\sharp)^{d-1} + \cdots + c_d^\sharp && \text{mod } p\mathcal{O}_C \\ &= (\alpha^d + c_1\alpha^{d-1} + \cdots + c_d)^\sharp && \text{mod } p\mathcal{O}_C \\ &= g(\alpha)^\sharp = 0 \end{aligned}$$

where the third identity follows from Proposition 2.1.9 in Chapter III. Hence we complete the proof by taking $z = \alpha^\sharp$. □

Proposition 1.1.6. Every untilt of F is algebraically closed.

Proof. Let C be an untilt of F, and let $f(x)$ an arbitrary monic irreducible polynomial of degree d over C. We wish to show that $f(x)$ has a root in C. We may replace $f(x)$ by $p^{nd} f(x/p^n)$ for sufficiently large n to assume that $f(x)$ is a polynomial over $\mathcal{O}_C$. Let us set $y_0 := 0$ so that we have $\nu_C(f(y_0)) = \nu_C(f(0)) \geq 0$. By Proposition 1.1.5 we can inductively construct a sequence (y_n) in C with

$$\nu_C(y_{n-1} - y_n) \geq (n-1)\nu_C(p)/d \qquad \text{and} \qquad \nu_C(f(y_n)) \geq n\nu_C(p) \qquad \text{for each } n \geq 1.$$

Then the sequence (y_n) is Cauchy by construction, and thus converges to an element $y \in C$. Hence we find

$$f(y) = f\left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} f(y_n) = 0,$$

thereby deducing the desired assertion. □

Remark. In order to avoid a circular reasoning, we should not deduce Proposition 1.1.6 as a special case of the tilting equivalence for perfectoid fields. In fact, the only known proof of the tilting equivalence (due to Scholze) is based on Proposition 1.1.6.

Corollary 1.1.7. For every untilt C of F, the associated sharp map is surjective.

We now aim to parametrize all untilts of F by certain principal ideals of A_{inf}.

Definition 1.1.8. Let C_1 and C_2 be untilts of F with continuous isomorphisms $\iota_1 : F \simeq C_1^\flat$ and $\iota_2 : F \simeq C_2^\flat$. We say that C_1 and C_2 are *equivalent* if there exists a continuous isomorphism $C_1 \simeq C_2$ such that the induced isomorphism $C_1^\flat \simeq C_2^\flat$ fits into the commutative diagram

$$\begin{array}{ccc} C_1^\flat & \xrightarrow{\sim} & C_2^\flat \\ & {\scriptstyle \iota_1}\nwarrow^{\sim} \quad \nearrow^{\sim}{\scriptstyle \iota_2} & \\ & F & \end{array}.$$

Example 1.1.9. Proposition 2.1.13 in Chapter III implies that the trivial untilt of F represents a unique equivalence class of characteristic p untilts of F.

PROPOSITION 1.1.10. Let C be a perfectoid field.

(1) Every continuous isomorphism $F \simeq C^\flat$ induces an isomorphism $\mathcal{O}_F/\varpi\mathcal{O}_F \simeq \mathcal{O}_C/p\mathcal{O}_C$ for some $\varpi \in \mathfrak{m}_F$.
(2) Every isomorphism $\mathcal{O}_F/\varpi\mathcal{O}_F \simeq \mathcal{O}_C/p\mathcal{O}_C$ for some $\varpi \in \mathfrak{m}_F$ uniquely lifts to a continuous isomorphism $F \simeq C^\flat$.

PROOF. Let us first consider the statement (1). We regard C as an untilt of F with the given continuous isomorphism $F \simeq C^\flat$. Then Proposition 2.1.10 in Chapter III yields an element $\varpi \in F$ with $\nu_F(\varpi) = \nu_C(p) > 0$. Moreover, the continuous isomorphism $F \simeq C^\flat$ restricts to an isomorphism of valuation rings $\mathcal{O}_F \simeq \mathcal{O}_{C^\flat}$. Let us now consider the map

$$\mathcal{O}_F \xrightarrow{c \mapsto c^\sharp} \mathcal{O}_C \twoheadrightarrow \mathcal{O}_C/p\mathcal{O}_C$$

where the second arrow is the natural projection. This map is a ring homomorphism as noted in Chapter III, Proposition 2.1.9, and is surjective by Lemma 2.1.8 in Chapter III. In addition, the kernel consists precisely of the elements $c \in \mathcal{O}_F$ with $\nu_C(c^\sharp) \geq \nu_C(p)$, or equivalently $\nu_F(c) \geq \nu_F(\varpi)$. Hence we have an induced isomorphism $\mathcal{O}_F/\varpi\mathcal{O}_F \simeq \mathcal{O}_C/p\mathcal{O}_C$ as asserted.

It remains to prove the statement (2). Since F is isomorphic to its tilt as noted in Example 1.1.9, we have an identification $\mathcal{O}_F \cong \mathcal{O}_{F^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_F$. Hence every isomorphism $\mathcal{O}_F/\varpi\mathcal{O}_F \simeq \mathcal{O}_C/p\mathcal{O}_C$ for some $\varpi \in \mathfrak{m}_F$ uniquely gives rise to an isomorphism

$$\mathcal{O}_F \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_F/\varpi\mathcal{O}_F \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p\mathcal{O}_C \cong \mathcal{O}_{C^\flat}$$

where the first and the third isomorphisms are given by Proposition 2.1.6 in Chapter III, and in turn lifts to a continuous isomorphism $F \simeq C^\flat$. $\square$

Definition 1.1.11. We say that an element $\xi \in A_{\text{inf}}$ is *primitive (of degree* 1*)* if it has the form $\xi = [\varpi] - up$ for some $\varpi \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$. We say that a primitive element of A_{inf} is *nondegenerate* if it is not divisible by p.

PROPOSITION 1.1.12. Let ξ be an element in A_{inf} with the Teichmüller expansion $\xi = \sum [c_n]p^n$.

(1) The element ξ is primitive if and only if we have $\nu_F(c_0) > 0$ and $\nu_F(c_1) = 0$.
(2) If ξ is primitive, every unit multiple of ξ in A_{inf} is primitive.

PROOF. The first statement is straightforward to verify by writing $\xi = [c_0] + p \sum [c_{n+1}^{1/p}]p^n$. The second statement then follows by the explicit multiplication rule for A_{inf}. $\square$

Proposition 1.1.13. Let ξ be a nondegenerate primitive element in A_{inf}. The ring $A_{\text{inf}}/\xi A_{\text{inf}}$ is p-torsion free and p-adically complete.

Proof. We first verify that $A_{\text{inf}}/\xi A_{\text{inf}}$ is p-torsion free. Consider an element $a \in A_{\text{inf}}$ such that pa is divisible by ξ. We wish to show that a is divisible by ξ. Let us write $pa = \xi b$ for some $b \in A_{\text{inf}}$. Then we have $b \in pA_{\text{inf}}$ since ξ has a nonzero image in $A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$. Therefore we may write $b = pb'$ for some $b' \in A_{\text{inf}}$ and obtain an identity $pa = p\xi b'$, which in turn yields $a = \xi b'$ as desired.

Let us now prove that $A_{\text{inf}}/\xi A_{\text{inf}}$ is p-adically complete. Denote by $\widehat{A_{\text{inf}}/\xi A_{\text{inf}}}$ the p-adic completion of $A_{\text{inf}}/\xi A_{\text{inf}}$. Since A_{inf} is p-adically complete, the projection $A_{\text{inf}} \twoheadrightarrow A_{\text{inf}}/\xi A_{\text{inf}}$ induces a surjective ring homomorphism

$$A_{\text{inf}} \twoheadrightarrow \widehat{A_{\text{inf}}/\xi A_{\text{inf}}} \tag{1.2}$$

by a general fact as stated in [**Sta**, Tag 0315]. It suffices to show that the kernel of this map is ξA_{inf}. Under the identification

$$\widehat{A_{\text{inf}}/\xi A_{\text{inf}}} = \varprojlim_n (A_{\text{inf}}/\xi A_{\text{inf}})/((p^n A_{\text{inf}} + \xi A_{\text{inf}})/\xi A_{\text{inf}}) \cong \varprojlim_n A_{\text{inf}}/(p^n A_{\text{inf}} + \xi A_{\text{inf}})$$

the map (1.2) coincides with the natural map

$$A_{\text{inf}} \twoheadrightarrow \varprojlim_n A_{\text{inf}}/(p^n A_{\text{inf}} + \xi A_{\text{inf}}).$$

The kernel of this map is $\bigcap_{n=1}^{\infty}(p^n A_{\text{inf}} + \xi A_{\text{inf}})$, which clearly contains ξA_{inf}. Hence we only need to show $\bigcap_{n=1}^{\infty}(p^n A_{\text{inf}} + \xi A_{\text{inf}}) \subseteq \xi A_{\text{inf}}$. Consider an arbitrary element $u \in \bigcap_{n=1}^{\infty}(p^n A_{\text{inf}} + \xi A_{\text{inf}})$. Let us choose sequences (a_n) and (b_n) in A_{inf} with $u = p^n a_n + \xi b_n$ for each $n \geq 1$. Then we have $p^n(a_n - pa_{n+1}) = \xi(b_{n+1} - b_n)$ for every $n \geq 1$. Since ξ has a nonzero image in $A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$, each $b_{n+1} - b_n$ must be divisible by p^n. Hence the sequence (b_n) converges to an element $b \in A_{\text{inf}}$ by the p-adic completeness of A_{inf}. As a result we find

$$u = \lim_{n\to\infty}(p^n a_n + \xi b_n) = \lim_{n\to\infty} p^n a_n + \xi \lim_{n\to\infty} b_n = \xi b,$$

thereby completing the proof. □

Definition 1.1.14. For every primitive element $\xi \in A_{\text{inf}}$, we refer to the natural projection $\theta_\xi : A_{\text{inf}} \twoheadrightarrow A_{\text{inf}}/\xi A_{\text{inf}}$ as the *untilt map* associated to ξ.

Lemma 1.1.15. Let ξ be a nondegenerate primitive element in A_{inf}.

(1) For every nonzero $c \in \mathcal{O}_F$, some power of p is divisible by $\theta_\xi([c])$ in $A_{\text{inf}}/\xi A_{\text{inf}}$.
(2) For every $m \in \mathfrak{m}_F$, some power of $\theta_\xi([m])$ is divisible by p in $A_{\text{inf}}/\xi A_{\text{inf}}$.

Proof. Let us write $\xi = [\varpi] - pu$ for some $\varpi \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$. For every nonzero $c \in \mathcal{O}_F$ we may write $\varpi^i = cc'$ for some $i > 0$ and $c' \in \mathcal{O}_F$, and consequently find

$$p^i = \big(\theta_\xi(u^{-1})\theta_\xi(up)\big)^i = \theta_\xi(u)^{-i}\theta_\xi([\varpi])^i = \theta_\xi(u)^{-i}\theta_\xi([c])\theta_\xi([c']).$$

Similarly, for every $m \in \mathfrak{m}_F$ we may write $m^j = \varpi \cdot b$ for some $j > 0$ and $b \in \mathcal{O}_F$, and consequently find

$$\theta_\xi([m])^j = \theta_\xi([\varpi])\theta_\xi([b]) = \theta_\xi(pu)\theta_\xi([b]) = p\theta_\xi(u)\theta_\xi([b]).$$

We thus deduce the desired assertions. □

PROPOSITION 1.1.16. Let ξ be a nondegenerate primitive element in A_{inf}. Take arbitrary elements $c, c' \in \mathcal{O}_F$. Then c divides c' in $\mathcal{O}_F$ if and only if $\theta_\xi([c])$ divides $\theta_\xi([c'])$ in $A_{\mathrm{inf}}/\xi A_{\mathrm{inf}}$.

PROOF. If c divides c' in $\mathcal{O}_F$, then $\theta_\xi([c])$ divides $\theta_\xi([c'])$ in $A_{\mathrm{inf}}/\xi A_{\mathrm{inf}}$ by the multiplicativity of the Teichmüller lift and the map θ_ξ. Let us now assume that c does not divide c' in $\mathcal{O}_F$. We wish to show that $\theta_\xi([c])$ does not divide $\theta_\xi([c'])$ in $A_{\mathrm{inf}}/\xi A_{\mathrm{inf}}$. Suppose for contradiction that there exists an element $a \in A_{\mathrm{inf}}/\xi A_{\mathrm{inf}}$ with $\theta_\xi([c']) = \theta_\xi([c])a$. Since we have $\nu_F(c) > \nu_F(c')$ by assumption, there exists some $m \in \mathfrak{m}_F$ with $c = mc'$. We thus find

$$\theta_\xi([c']) = \theta_\xi([c])a = \theta_\xi([c'])\theta_\xi([m])a. \tag{1.3}$$

Moreover, c' is not zero as it is not divisible by c. Hence by Lemma 1.1.15 we may write $p^n = \theta_\xi([c'])b$ for some $n > 0$ and $b \in A_{\mathrm{inf}}/\xi A_{\mathrm{inf}}$. Then by (1.3) we find $p^n = p^n\theta_\xi([m])a$, which in turn yields $\theta_\xi([m])a = 1$ since p is not a zero divisor in $A_{\mathrm{inf}}/\xi A_{\mathrm{inf}}$ by Proposition 1.1.13. However, this is impossible because the image of $\theta_\xi([m])$ under the natural map $A_{\mathrm{inf}}/\xi A_{\mathrm{inf}} \twoheadrightarrow A_{\mathrm{inf}}/(\xi A_{\mathrm{inf}} + pA_{\mathrm{inf}})$ is nilpotent by Lemma 1.1.15. □

PROPOSITION 1.1.17. Let ξ be a nondegenerate primitive element in A_{inf}. Every $a \in A_{\mathrm{inf}}/\xi A_{\mathrm{inf}}$ is a unit multiple of $\theta_\xi([c])$ for some $c \in \mathcal{O}_F$, which is uniquely determined up to unit multiple.

PROOF. Let us first assume that a is a unit multiple of $\theta_\xi([c_1])$ and $\theta_\xi([c_2])$ for some $c_1, c_2 \in \mathcal{O}_F$. Then $\theta_\xi([c_1])$ and $\theta_\xi([c_2])$ divide each other. Hence Proposition 1.1.16 implies that c_1 and c_2 divide each other, which means that c_1 and c_2 are unit multiples of each other.

It remains to show that a is a unit multiple of $\theta_\xi([c])$ for some $c \in \mathcal{O}_F$. We may assume $a \neq 0$ as the assertion is obvious for $a = 0$. By Proposition 1.1.13 we can write $a = p^n a'$ for some $n \geq 0$ and $a' \in A_{\mathrm{inf}}/\xi A_{\mathrm{inf}}$ such that a' is not divisible by p. Let us write $\xi = [\varpi] - up$ for some $\varpi \in \mathfrak{m}_F$ and $u \in A_{\mathrm{inf}}^\times$. Then we have

$$a = p^n a' = \left(\theta_\xi(u^{-1})\theta_\xi(up)\right)^n a' = \theta_\xi(u)^{-1}\theta_\xi([\varpi])^n a'.$$

Hence we may replace a by a' to assume that a is not divisible by p.

We have a natural isomorphism

$$A_{\mathrm{inf}}/(\xi A_{\mathrm{inf}} + pA_{\mathrm{inf}}) = A_{\mathrm{inf}}/([\varpi]A_{\mathrm{inf}} + pA_{\mathrm{inf}}) \cong \mathcal{O}_F/\varpi\mathcal{O}_F.$$

In addition, the map θ_ξ gives rise to a commutative diagram

$$\begin{array}{ccc} A_{\mathrm{inf}} & \xrightarrow{\theta_\xi} & A_{\mathrm{inf}}/\xi A_{\mathrm{inf}} \\ \downarrow & & \downarrow \\ \mathcal{O}_F \cong A_{\mathrm{inf}}/pA_{\mathrm{inf}} & \twoheadrightarrow & A_{\mathrm{inf}}/(\xi A_{\mathrm{inf}} + pA_{\mathrm{inf}}) \cong \mathcal{O}_F/\varpi\mathcal{O}_F \end{array} \tag{1.4}$$

where the surjectivity of the bottom middle arrow follows from the surjectivity of the other arrows. Choose an element $c \in \mathcal{O}_F$ whose image under the bottom middle arrow coincides with the image of a under the second vertical arrow. Then c is not divisible by ϖ since a is not divisible by p. Therefore we may write $\varpi = cm$ for some $m \in \mathfrak{m}_F$ and obtain

$$p = \theta_\xi(u^{-1})\theta_\xi(up) = \theta_\xi(u)^{-1}\theta_\xi([\varpi]) = \theta_\xi(u)^{-1}\theta_\xi([c])\theta_\xi([m]).$$

Now the diagram (1.4) yields an element $b \in A_{\mathrm{inf}}/\xi A_{\mathrm{inf}}$ with

$$a = \theta_\xi([c]) + pb = \theta_\xi([c]) + b\theta_\xi(u)^{-1}\theta_\xi([c])\theta_\xi([m]) = \theta_\xi([c])\left(1 + b\theta_\xi(u)^{-1}\theta_\xi([m])\right).$$

We thus complete the proof by observing that $1 + b\theta_\xi(u)^{-1}\theta_\xi([m])$ is a unit in $A_{\mathrm{inf}}/\xi A_{\mathrm{inf}}$ with

$$\left(1 + b\theta_\xi(u)^{-1}\theta_\xi([m])\right)^{-1} = 1 - \left(b\theta_\xi(u)^{-1}\theta_\xi([m])\right) + \left(b\theta_\xi(u)^{-1}\theta_\xi([m])\right)^2 - \cdots$$

where the infinite sum converges by Proposition 1.1.13 and Lemma 1.1.15. □

PROPOSITION 1.1.18. Let ξ be a primitive element in A_{inf}, and let C_ξ denote the fraction field of $A_{\text{inf}}/\xi A_{\text{inf}}$. Then C_ξ is an untilt of F with the valuation ring $\mathcal{O}_{C_\xi} = A_{\text{inf}}/\xi A_{\text{inf}}$ and a continuous isomorphism $\iota : F \simeq C_\xi^\flat$ induced by the canonical isomorphism

$$\mathcal{O}_F/\varpi\mathcal{O}_F \cong A_{\text{inf}}/([\varpi]A_{\text{inf}} + pA_{\text{inf}}) = A_{\text{inf}}/(\xi A_{\text{inf}} + pA_{\text{inf}}) \cong \mathcal{O}_{C_\xi}/pC_\xi. \qquad (1.5)$$

where ϖ denotes the image of ξ under the natural map $A_{\text{inf}} \twoheadrightarrow A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$. Moreover, each element $c \in \mathcal{O}_F$ maps to $\theta_\xi([c])$ under the sharp map associated to C_ξ.

PROOF. Let us write $\xi = [\varpi] - up$ with $\varpi \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$. We also let $\mathcal{O}$ denote the ring $A_{\text{inf}}/\xi A_{\text{inf}}$. If ϖ is zero, then we have a natural isomorphism

$$\mathcal{O} = A_{\text{inf}}/\xi A_{\text{inf}} \cong A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$$

which implies that C_ξ represents the trivial untilt of F as noted in Example 1.1.9. We henceforth assume $\varpi \neq 0$.

We assert that $\mathcal{O} = A_{\text{inf}}/\xi A_{\text{inf}}$ is an integral domain. Suppose for contradiction that there exist nonzero elements $a, b \in \mathcal{O}$ with $ab = 0$. By Proposition 1.1.17 we may write $a = \theta_\xi([c])u$ for some nonzero $c \in \mathcal{O}_F$ and $u \in \mathcal{O}^\times$. In addition, by Lemma 1.1.15 we have $\theta_\xi([c])w = p^n$ for some $n > 0$ and $w \in \mathcal{O}$. Therefore we obtain an identity

$$0 = abw = \theta_\xi([c])wub = p^n ub,$$

which yields a desired contradiction by Proposition 1.1.13.

By Proposition 1.1.17 we can define a nonnegative real-valued function ν on $\mathcal{O}^\times$ which maps each $y \in \mathcal{O}^\times$ to $\nu_F(z)$ where z is an element in $\mathcal{O}_F$ such that y is a unit multiple of $\theta_\xi([z])$. Then by construction ν is a multiplicative homomorphism whose image contains the image of ν_F. In addition, for any $y_1, y_2 \in \mathcal{O}^\times$ with $\nu(y_1) \geq \nu(y_2)$ we find by Proposition 1.1.16 that y_1 is divisible y_2 in $\mathcal{O}$, and consequently obtain

$$\nu(y_1 + y_2) = \nu((y_1/y_2 + 1)y_2) = \nu(y_1/y_2 + 1) + \nu(y_2) \geq \nu(y_2) = \min(\nu(y_1), \nu(y_2)).$$

Therefore we deduce that ν is a nondiscrete valuation on $\mathcal{O}$.

We can uniquely extend ν to a valuation on C_ξ, which we also denote by ν. For every $x \in C_\xi$ we write $x = y_1/y_2$ for some $y_1, y_2 \in \mathcal{O}$ and find by Proposition 1.1.16 that $\nu(x) = \nu(y_1) - \nu(y_2)$ is nonnegative if and only if y_1 is divisible by y_2 in $\mathcal{O}$. Hence we deduce that $\mathcal{O}$ is indeed the valuation ring of C_ξ.

Since the p-th power map is surjective on $\mathcal{O}_F/\varpi\mathcal{O}_F$, it is also surjective on $\mathcal{O}_{C_\xi}/p\mathcal{O}_{C_\xi}$ by the isomorphism (1.5). In addition, from the identity

$$p = \theta_\xi(u^{-1})\theta_\xi(up) = \theta_\xi(u)^{-1}\theta_\xi([\varpi])$$

we find $\nu(p) = \nu_F(\varpi) > 0$. Hence C_ξ has residue characteristic p. Furthermore, Proposition 1.1.13 implies that C_ξ is complete with respect to the valuation ν. Therefore we deduce that C_ξ is a perfectoid field.

By Proposition 1.1.10 (and its proof) the isomorphism (1.5) uniquely lifts to an isomorphism

$$\mathcal{O}_F \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_F/\varpi\mathcal{O}_F \cong \varprojlim_{x \mapsto x^p} A_{\text{inf}}/(\xi A_{\text{inf}} + pA_{\text{inf}}) \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_{C_\xi}/p\mathcal{O}_{C_\xi} \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_{C_\xi} = \mathcal{O}_{C_\xi^\flat}$$

where the first and the third isomorphisms are given by Proposition 2.1.6 in Chapter III, and in turn lifts to a continuous isomorphism $F \simeq C_\xi^\flat$. Moreover, it is straightforward to verify that each element $c \in \mathcal{O}_F$ maps to $(\theta_\xi([c^{1/p^n}]) \in \mathcal{O}_{C_\xi^\flat}$ under the above isomorphism, and consequently maps to $\theta_\xi([c])$ under the sharp map associated to C_ξ. Therefore we complete the proof. $\square$

PROPOSITION 1.1.19. Let C be an untilt of F.

(1) There exists a surjective ring homomorphism $\theta_C : A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$ with
$$\theta_C\left(\sum [c_n]p^n\right) = \sum c_n^\sharp p^n \qquad \text{for every } c_n \in \mathcal{O}_F.$$

(2) Every primitive element in $\ker(\theta_C)$ generates $\ker(\theta_C)$.

PROOF. Since C is algebraically closed as noted in Proposition 1.1.6, all results from the first part of §2.2 in Chapter III remain valid with C in place of $\mathbb{C}_K$. In particular, the statement (1) is merely a restatement of Proposition 2.2.3 in Chapter III. Furthermore, Proposition 2.2.11 in Chapter III implies that $\ker(\theta_C)$ is generated by a primitive element $\xi_C := [p^\flat] - p \in A_{\text{inf}}$ where $p^\flat$ denotes an element in $\mathcal{O}_F$ with $(p^\flat)^\sharp = p$.

Let us now consider an arbitrary primitive element $\xi \in \ker(\theta_C)$. The map θ_C induces a surjective map $\widetilde{\theta_\xi} : A_{\text{inf}}/\xi A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$. Then $\ker(\widetilde{\theta_\xi})$ is a non-maximal prime ideal as $\mathcal{O}_C$ is an integral domain but not a field. Moreover, $\ker(\widetilde{\theta_\xi})$ is a principal ideal generated by the image of ξ_C. Since A_{inf}/ξ is a valuation ring by Proposition 1.1.18, we find $\ker(\widetilde{\theta_\xi}) = 0$ and consequently deduce that ξ generates $\ker(\theta_C)$. □

Remark. In the last sentence, we used an elementary fact that every nonzero principal prime ideal of a valuation ring is maximal.

Definition 1.1.20. Given an untilt C of F, we refer to the ring homomorphism θ_C constructed in Proposition 1.1.19 as the *untilt map* of C.

THEOREM 1.1.21 (Kedlaya-Liu [**KL15**], Fontaine [**Fon13**]). There is a bijection
$$\{\text{ equivalence classes of untilts of } F\,\} \stackrel{\sim}{\longrightarrow} \{\text{ ideals of } A_{\text{inf}} \text{ generated by a primitive element }\}$$
which maps each untilt C of F to $\ker(\theta_C)$.

PROOF. We first verify that the association is surjective. Consider an arbitrary primitive element $\xi \in A_{\text{inf}}$. By Proposition 1.1.18 it gives rise to an untilt C_ξ of F such that each $c \in \mathcal{O}_F$ maps to $\theta_\xi([c])$ under the associated sharp map. Hence Lemma 2.3.1 from Chapter II implies that the maps θ_ξ and θ_{C_ξ} coincide, thereby yielding $\xi A_{\text{inf}} = \ker(\theta_\xi) = \ker(\theta_{C_\xi})$.

It remains to show that the association is injective. Let C be an arbitrary untilt of F with a continuous isomorphism $\iota : F \simeq C^\flat$. Choose a primitive element $\omega \in \ker(\theta_C)$, which gives rise to an untilt C_ω of F with a continuous isomorphism $\iota_\omega : F \simeq C_\omega^\flat$ by Proposition 1.1.18. It suffices to show that C and C_ω are equivalent. The map θ_C induces an isomorphism $\mathcal{O}_{C_\omega} = A_{\text{inf}}/\omega A_{\text{inf}} \simeq \mathcal{O}_C$, which extends to an isomorphism $C_\omega \simeq C$. Let f denote the induced map $C_\omega^\flat \simeq C^\flat$. Then by Proposition 1.1.10 and Proposition 1.1.18 the map $f \circ \iota_\omega$ yields an isomorphism
$$\mathcal{O}_F/\varpi\mathcal{O}_F \cong A_{\text{inf}}/(pA_{\text{inf}} + \omega A_{\text{inf}}) = \mathcal{O}_{C_\omega}/p\mathcal{O}_{C_\omega} \simeq \mathcal{O}_C/p\mathcal{O}_C \tag{1.6}$$
where ϖ denotes the image of ω in $A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$. For every $c \in \mathcal{O}_F$, this isomorphism maps the image of c in $\mathcal{O}_F/\varpi\mathcal{O}_F$ to the image of $\theta_C([c]) = c^\sharp$ in $\mathcal{O}_C/p\mathcal{O}_C$. This implies that an element $c \in \mathcal{O}_F$ is divisible by ϖ if and only if $c^\sharp$ is divisible by p, and consequently yields $\nu_F(\varpi) = \nu_C(p)$. Then the proof of Proposition 1.1.10 shows that the isomorphism (1.6) is also induced by ι. Therefore the second part of Proposition 1.1.10 yields $f \circ \iota_\omega = \iota$, which means that C and C_ω are equivalent as desired. □

Remark. The first paragraph of our proof shows that there is no conflict between Definition 1.1.14 and Definition 1.1.20.

1.2. The schematic Fargues-Fontaine curve

The main goal of this subsection is to describe the construction of the Fargues-Fontaine curve as a scheme. For the rest of this chapter, we fix a nonzero element $\varpi \in \mathfrak{m}_F$. We also denote by $Y_F = Y$ the set of equivalence classes of characteristic 0 untilts of F.

Definition 1.2.1. Let C be an untilt of F. We define the *associated absolute value* on C by

$$|x|_C := p^{-\nu_C(x)} \qquad \text{for every } x \in C,$$

and write $|C| := \{ |x|_C : x \in C \}$ for the associated absolute value group. If $C = F$ is the trivial untilt of F, we often drop the subscript to ease the notation.

Remark. Thus far we have been using valuations to describe the topology on valued fields, because valuations are convenient for topological arguments involving algebraic objects such as p-adic representations and period rings. From now on, we will use absolute values to describe the topology on perfectoid fields, because the objects of our interest will be very much analytic in nature.

Example 1.2.2. Let C be an untilt of F. Theorem 1.1.21 yields a primitive element $\xi \in A_{\text{inf}}$ which generates $\ker(\theta_C)$. If we write $\xi = [m] - up$ for some $m \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$, we have

$$|p|_C = \left|\theta_C(u)^{-1}\theta_C([m])\right|_C = |\theta_C([m])|_C = \left|m^\sharp\right|_C = |m|.$$

PROPOSITION 1.2.3. We have an identification

$$A_{\text{inf}}[1/p, 1/[\varpi]] = \left\{ \sum [c_n]p^n \in W(F)[1/p] : |c_n| \text{ bounded} \right\}.$$

In particular, the ring $A_{\text{inf}}[1/p, 1/[\varpi]]$ does not depend on our choice of ϖ.

PROOF. Consider an arbitrary element $f = \sum [c_n]p^n \in W(F)[1/p]$. Then we have $f \in A_{\text{inf}}[1/p, 1/[\varpi]]$ if and only if there exists some $i > 0$ with $[\varpi^i]f = \sum [c_n\varpi^i]p^n \in A_{\text{inf}}[1/p]$, or equivalently $|c_n| \leq \left|\varpi^{-i}\right|$ for all n. □

Definition 1.2.4. Let y be an element of Y, represented by an untilt C of F.

(1) We define the *absolute value* of y by $|y| := |p|_C$.
(2) For every $f = \sum [c_n]p^n \in A_{\text{inf}}[1/p, 1/[\varpi]]$, we define its *value at* y by

$$f(y) := \widetilde{\theta_C}(f) = \sum c_n^\sharp p^n$$

where $\widetilde{\theta_C} : A_{\text{inf}}[1/p, 1/[\varpi]] \longrightarrow C$ is the ring homomorphism which extends the untilt map $\theta_C : A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$.

Remark. A useful heuristic idea for understanding the construction and the structure of the Fargues-Fontaine curve is that the set Y behaves in many aspects as the punctured unit disk $\mathbb{D}^* := \{ z \in \mathbb{C} : 0 < |z| < 1 \}$ in the complex plane. Here we present a couple of analogies between Y and $\mathbb{D}^*$.

(1) For each $y \in Y$ represented by an untilt C of F, its absolute value $|y| = |p|_C$ is a real number between 0 and 1. This is an analogue of the fact that every element $z \in \mathbb{D}^*$ has an absolute value between 0 and 1.
(2) Every element in $A_{\text{inf}}[1/p, 1/[\varpi]]$ is a "Laurent series in the variable p" with bounded coefficients, and gives rise to a function on Y as described in Definition 1.2.4. This is an analogue of the fact that every Laurent series $\sum a_n z^n$ over $\mathbb{C}$ with bounded coefficients defines a holomorphic function on $\mathbb{D}^*$.

LEMMA 1.2.5. Let $f = \sum [c_n]p^n$ be a nonzero element in $A_{\inf}[1/p, 1/[\varpi]]$, and let ρ be a real number with $0 < \rho < 1$. Then $\sup_{n\in\mathbb{Z}}(|c_n|\,\rho^n)$ exists and is attained by finitely many values of n.

PROOF. Let us take an integer n_0 with $c_{n_0} \neq 0$. Proposition 1.2.3 implies that there exists an integer $l > 0$ with $|c_n|\,\rho^n < |c_{n_0}|\,\rho^{n_0}$ for all $n > l$. In addition, there exists an integer $k < 0$ with $c_n = 0$ for all $n < k$. Therefore $\sup_{n\in\mathbb{Z}}(|c_n|\,\rho^n) = \sup_{k<n<l}(|c_n|\,\rho^n)$ exists and can only be attained by an integer n with $k < n < l$. □

Definition 1.2.6. Let ρ be a real number with $0 < \rho < 1$.

(1) We define the *Gauss ρ-norm* on $A_{\inf}[1/p, 1/[\varpi]]$ by

$$\left|\sum [c_n]p^n\right|_\rho := \sup_{n\in\mathbb{Z}}(|c_n|\,\rho^n).$$

(2) Given an element $f = \sum [c_n]p^n \in A_{\inf}[1/p, 1/[\varpi]]$, we say that ρ is *generic* for f if there exists a unique $n \in \mathbb{Z}$ with $|f|_\rho = |c_n|\,\rho^n$.

LEMMA 1.2.7. Let f be an element in $A_{\inf}[1/p, 1/[\varpi]]$. The set

$$S_f := \{\, \rho \in (0,1) : \rho \text{ is generic for } f \,\}$$

is dense in the interval $(0, 1)$.

PROOF. If $\rho \in (0, 1)$ is not generic for f, then by Lemma 1.2.5 there exist some distinct integers m and n with $|f|_\rho = |c_m|\,\rho^m = |c_n|\,\rho^n$, which yields $\rho = (|c_m| / |c_n|)^{1/(n-m)}$. We thus deduce that the complement of S_f in $(0, 1)$ is countable, thereby obtaining the assertion. □

LEMMA 1.2.8. Let y be an element in Y represented by an untilt C of F. For every $f \in A_{\inf}[1/p, 1/[\varpi]]$ we have $|f(y)|_C \leq |f|_{|y|}$ with equality if $|y|$ is generic for f.

PROOF. Let us write $f = \sum [c_n]p^n$ with $c_n \in F$. Then we have

$$|f(y)|_C = \left|\sum c_n^\sharp p^n\right|_C \leq \sup_{n\in\mathbb{Z}}\left(\left|c_n^\sharp\right|_C \cdot |p|_C^n\right) = \sup_{n\in\mathbb{Z}}(|c_n| \cdot |y|^n) = |f|_{|y|}\,.$$

It is evident that the inequality above becomes an equality if $|y|$ is generic for f. □

PROPOSITION 1.2.9. For every positive real number $\rho < 1$, the Gauss ρ-norm on $A_{\inf}[1/p, 1/[\varpi]]$ is a multiplicative norm.

PROOF. Let f and g be arbitrary elements in $A_{\inf}[1/p, 1/[\varpi]]$. We wish to show

$$|f+g|_\rho \leq \max(|f|_\rho\,, |g|_\rho) \qquad \text{and} \qquad |fg|_\rho = |f|_\rho\,|g|_\rho\,.$$

Since $|F|$ is dense in the set of nonnegative real numbers, Lemma 1.2.7 implies that the set

$$S := \{\, \tau \in (0,1) \cap |F| : \tau \text{ is generic for } f, g, f+g, \text{ and } fg \,\}$$

is dense in the interval $(0, 1)$. Hence we write $\rho = \lim_{n\to\infty} \tau_n$ for some (τ_n) in S to assume $\rho \in S$.

Take an element $c \in \mathfrak{m}_F$ with $|c| = \rho$. Then $\xi := [c] - p \in A_{\inf}$ is a nondegenerate primitive element, and thus gives rise to an element $y \in Y$ with $|y| = \rho$ by Proposition 1.1.13, Theorem 1.1.21, and Example 1.2.2. Then by Lemma 1.2.8 we find

$$\begin{aligned} |f+g|_\rho &= |f(y)+g(y)|_C \leq \max(|f(y)|_C\,, |g(y)|_C) = \max(|f|_\rho\,, |g|_\rho), \\ |fg|_\rho &= |f(y)g(y)|_C = |f(y)|_C\,|g(y)|_C = |f|_\rho\,|g|_\rho\,. \end{aligned}$$

Therefore we complete the proof. □

Definition 1.2.10. Let $[a, b]$ be a closed subinterval of $(0, 1)$. We write

$$Y_{[a,b]} := \{\, y \in Y : a \le |y| \le b \,\},$$

and define the *ring of holomorphic functions on* $Y_{[a,b]}$, denoted by $B_{[a,b]}$, to be the completion of $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ with respect to the Gauss a-norm and the Gauss b-norm.

Lemma 1.2.11. Let $[a, b]$ be a closed subinterval of $(0, 1)$, and let f be an element in $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$. We have $|f|_\rho \le \sup(|f|_a, |f|_b)$ for all $\rho \in [a, b]$.

Proof. Let us write $f = \sum [c_n] p^n$ for some $c_n \in F$. Then we have

$$\begin{aligned} |c_n|\, \rho^n &\le |c_n|\, b^n \le |f|_b \qquad \text{for all } n \ge 0,\\ |c_n|\, \rho^n &\le |c_n|\, a^n \le |f|_a \qquad \text{for all } n < 0. \end{aligned}$$

Hence we deduce the desired assertion. □

Remark. Since $|F|$ is dense in $(0, \infty)$, we find $\sup\limits_{|y|=\rho} (|f(y)|_C) = |f|_\rho$ for all $\rho \in |F| \cap (0, 1)$ by Lemma 1.2.7 and Lemma 1.2.8. Hence we may regard Lemma 1.2.11 as an analogue of the maximum modulus principle for holomorphic functions on $\mathbb{D}^*$.

Proposition 1.2.12. Let $[a, b]$ be a closed subinterval of $(0, 1)$. The ring $B_{[a,b]}$ is the completion of $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ with respect to all Gauss ρ-norms with $\rho \in [a, b]$.

Proof. Lemma 1.2.11 implies that a sequence (f_n) in $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ is Cauchy with respect to the Gauss a-norm and the Gauss b-norm if and only if it is Cauchy with respect to the Gauss ρ-norm for all $\rho \in [a, b]$. □

Corollary 1.2.13. For any $a, b, a', b' \in \mathbb{R}$ with $[a, b] \subseteq [a', b'] \subseteq (0, 1)$, we have $B_{[a',b']} \subseteq B_{[a,b]}$.

Definition 1.2.14. We define the *ring of holomorphic functions on* Y by

$$B_F := \varprojlim B_{[a,b]}$$

where the transition maps are the natural inclusions given by Corollary 1.2.13. We often write B instead of B_F to ease the notation.

Remark. It is not hard to see that a formal sum $\sum [c_n] p^n$ with $c_n \in F$ converges in B if and only if it satisfies

$$\limsup_{n>0} |c_n|^{1/n} \le 1 \qquad \text{and} \qquad \lim_{n\to\infty} |c_{-n}|^{1/n} = 0.$$

This is an analogue of the fact that a Laurent series $\sum a_n z^n$ over $\mathbb{C}$ converges on $\mathbb{D}^*$ if and only if it satisfies

$$\limsup_{n>0} |a_n|^{1/n} \le 1 \qquad \text{and} \qquad \lim_{n\to\infty} |a_{-n}|^{1/n} = 0.$$

However, an arbitrary element in B may not admit a unique "Laurent series expansion" in p, whereas every holomorphic function on $\mathbb{D}^*$ admits a unique Laurent series expansion.

Lemma 1.2.15. Let $\eta : R_1 \longrightarrow R_2$ be a continuous homomorphism of normed rings.

(1) The map η uniquely extends to a continuous ring homomorphism $\widehat{\eta} : \widehat{R_1} \longrightarrow \widehat{R_2}$ where $\widehat{R_1}$ and $\widehat{R_2}$ respectively denote the completions of R_1 and R_2.
(2) The homomorphism $\widehat{\eta}$ is a homeomorphism if η is a homeomorphism.

Proof. This is an immediate consequence of an elementary fact from analysis. □

PROPOSITION 1.2.16. Let C be a characteristic 0 untilt of F. The untilt map θ_C uniquely extends to a surjective continuous open ring homomorphism $\widehat{\theta_C} : B \twoheadrightarrow C$.

PROOF. The map θ_C uniquely extends to a surjective ring homomorphism

$$\widetilde{\theta_C} : A_{\mathrm{inf}}[1/p, 1/[\varpi]] \twoheadrightarrow \mathcal{O}_C[1/p] = C.$$

Let us set $\rho := |p|_C$. Then $\widetilde{\theta_C}$ uniquely extends to a surjective continuous ring homomorphism $\widehat{\widehat{\theta_C}} : B_{[\rho,\rho]} \twoheadrightarrow C$ by Lemma 1.2.8 and Lemma 1.2.15. Moreover, $\widehat{\widehat{\theta_C}}$ is open by the open mapping theorem. Take $\widehat{\theta_C}$ to be the restriction of $\widehat{\widehat{\theta_C}}$ on B. By construction $\widehat{\theta_C}$ is a surjective continuous open map which extends $\widetilde{\theta_C}$. Since the uniqueness is evident by the continuity, we deduce the desired assertion. □

Definition 1.2.17. Let y be an element in Y, represented by an untilt C of F.

(1) We refer to the map $\widehat{\theta_C}$ given by Proposition 1.2.16 as the *evaluation map* at y.
(2) For every $f \in B$, we define its *value at* y by $f(y) := \widehat{\theta_C}(f)$.

PROPOSITION 1.2.18. The Frobenius automorphism of F uniquely lifts to a continuous automorphism φ on B.

PROOF. Let $\widetilde{\varphi_F}$ denote the Frobenius automorphism of $W(F)$. By construction we have

$$\widetilde{\varphi_F}\left(\sum [c_n]p^n\right) = \sum [c_n^p]p^n \qquad \text{for all } c_n \in F. \tag{1.7}$$

Then Proposition 1.2.3 implies that $\widetilde{\varphi_F}$ restricts to an automorphism on $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$. Moreover, by (1.7) we find

$$|\widetilde{\varphi_F}(f)|_{\rho^p} = |f|_\rho^p \qquad \text{for all } f \in A_{\mathrm{inf}}[1/p, 1/[\varpi]] \text{ and } \rho \in (0,1). \tag{1.8}$$

Consider an arbitrary closed interval $[a, b] \subseteq (0, 1)$, and choose a real number $r \in [a, b]$. By Lemma 1.2.15 and (1.8) the map $\widetilde{\varphi_F}$ on $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ uniquely extends to a continuous ring isomorphism $\varphi_{[r,r]} : B_{[r,r]} \simeq B_{[r^p,r^p]}$. In addition, the identity (1.8) implies that a sequence (f_n) in $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ is Cauchy with respect to the Gauss a-norm and the Gauss b-norm if and only if the sequence $(\widetilde{\varphi_F}(f_n))$ is Cauchy with respect to the Gauss a^p-norm and the Gauss b^p-norm. Since $\widetilde{\varphi_F}$ is bijective, we deduce that $\varphi_{[r,r]}$ restricts to a continuous ring isomorphism $\varphi_{[a,b]} : B_{[a,b]} \simeq B_{[a^p,b^p]}$ with an inverse given by the restriction of $\varphi_{[r,r]}^{-1}$ on $B_{[a^p,b^p]}$. It is evident by construction that $\varphi_{[a,b]}$ is an extension of $\widetilde{\varphi_F}$.

By our discussion in the preceding paragraph, the map $\widetilde{\varphi_F}$ on $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ extends to a continuous isomorphism

$$\varphi : B = \varprojlim B_{[a,b]} \simeq \varprojlim B_{[a^p,b^p]} = B.$$

Moreover, the uniqueness of φ is evident by the continuity. Therefore we obtain the desired assertion. □

Definition 1.2.19. We refer to the map φ constructed in Proposition 1.2.18 as the *Frobenius automorphism* of B, and define the *schematic Fargues-Fontaine curve* as the scheme

$$X_F := \mathrm{Proj}\left(\bigoplus_{n \geq 0} B^{\varphi = p^n}\right).$$

We often simply write X instead of X_F to ease the notation.

1.3. The adic Fargues-Fontaine curve

In this subsection, we describe another incarnation of the Fargues-Fontaine curve using the language of adic spaces developed by Huber in [**Hub93**] and [**Hub94**]. Our goal for this subsection is twofold: introducing a new perspective for the construction of the Fargues-Fontaine curve, and providing an exposition on some related theories. Our discussion will be cursory, as we won't use any results from this section in the subsequent sections.

Definition 1.3.1. Let R be a topological ring.

(1) We say that a subset S of R is *bounded* if for every open neighborhood U of 0 there exists an open neighborhood V of 0 with $VS \subseteq U$.
(2) We say that an element $f \in R$ is *power-bounded* if the set $\{ f^n : n \geq 0 \}$ is bounded, and denote by R° the subring of power-bounded elements in R.
(3) We say that R is a *Huber ring* if there exists an open subring R_0, called a *ring of definition*, on which the induced topology is generated by a finitely generated ideal.
(4) If R is a Huber ring, we say that R is *uniform* if R° is a ring of definition.

Example 1.3.2. We present some important examples of uniform Huber rings.

(1) Every ring R with the discrete topology is a uniform Huber ring with $R^\circ = R$, as its topology is generated by the zero ideal.
(2) Every nonarchimedean field L is a uniform Huber ring with $L^\circ = \mathcal{O}_L$, as the topology on $\mathcal{O}_L$ is generated by the ideal $m\mathcal{O}_L$ for any m in the maximal ideal.
(3) The ring A_{inf} is a uniform Huber ring with $A_{\text{inf}}^\circ = A_{\text{inf}}$ and the topology generated by the ideal $pA_{\text{inf}} + [\varpi]A_{\text{inf}}$.

Definition 1.3.3. A *Huber pair* is a pair (R, R^+) which consists of a Huber ring R and its open and integrally closed subring $R^+ \subseteq R^\circ$.

PROPOSITION 1.3.4. For every Huber ring R, the subring R° is open and integrally closed.

Definition 1.3.5. Let R be a topological ring.

(1) A map $v : R \longrightarrow T \cup \{ 0 \}$ for some totally ordered abelian group T is called a *continuous multiplicative valuation* if it satisfies the following properties:
   (i) $v(0) = 0$ and $v(1) = 1$.
   (ii) For all $r, s \in R$ we have $v(rs) = v(r)v(s)$ and $v(r+s) \leq \max(v(r), v(s))$.
   (iii) For every $\tau \in T$ the set $\{ r \in R : v(r) < \tau \}$ is open in R.
(2) We say that two continuous multiplicative valuations v and w on R are *equivalent* if there exists an isomorphism of totally ordered monoids $\delta : v(R) \cup \{ 0 \} \simeq w(R) \cup \{ 0 \}$ with $\delta(v(r)) = w(r)$ for all $r \in R$.
(3) We define the *valuation spectrum* of R, denoted by $\text{Spv}(R)$, to be the set of equivalence classes of continuous multiplicative valuations on R.
(4) Given $r \in R$ and $x \in \text{Spv}(R)$, we define the *value of* r *at* x by $|r(x)| := v(r)$ where v is any representative of x.

Remark. Our terminology in (1) slightly modifies Huber's original terminology *continuous valuation* in order to avoid any potential confusion after extensively using the term valuation in the additive notation.

PROPOSITION 1.3.6. Let v and w be continuous multiplicative valuations on a topological ring R. Then v and w are equivalent if and only if for all $r, s \in R$ the inequality $v(r) \leq v(s)$ amounts to the inequality $w(r) \leq w(s)$.

Definition 1.3.7. For a Huber pair (R, R^+), we define its *adic spectrum* by

$$\mathrm{Spa}(R, R^+) := \left\{ x \in \mathrm{Spv}(R) : |f(x)| \leq 1 \text{ for all } f \in R^+ \right\}$$

endowed with the topology generated by subsets of the form

$$\mathcal{U}(f/g) := \left\{ x \in \mathrm{Spa}(R, R^+) : |f(x)| \leq |g(x)| \neq 0 \right\} \qquad \text{for some } f, g \in R.$$

Example 1.3.8. We are particularly interested in the set

$$\mathcal{Y} := \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) \backslash \{ x \in \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) : |p[\varpi](x)| = 0 \},$$

which we call the *perfectoid punctured unit disk*. Let us describe two types of points on $\mathcal{Y}$.

Let y be an element in Y, represented by an untilt C of F. Consider a nonnegative real valued function v_y on A_{inf} defined by $v_y(f) := |f(y)|_C = |\theta_C(f)|_C$ for every $f \in A_{\mathrm{inf}}$. It is evident by construction that v_y is a continuous multiplicative valuation on A_{inf} with $v_y(f) \leq 1$ for all $f \in A_{\mathrm{inf}}$. In addition, we have $v_y(p) = |p|_C \neq 0$ and $v_y([\varpi]) = |\varpi| \neq 0$. Hence v_y gives rise to a point in $\mathcal{Y}$, which we denote by $\widetilde{y}$.

Let ρ be a real number with $0 < \rho < 1$. By Proposition 1.2.9 the Gauss ρ-norm on $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ restricts to a continuous multiplicative valuation on A_{inf} with $|f|_\rho \leq 1$ for all $f \in A_{\mathrm{inf}}$. In addition, we have $|p|_\rho = \rho \neq 0$ and $|[\varpi]|_\rho = |\varpi| \neq 0$. Hence the Gauss ρ-norm on $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ gives rise to a point in $\mathcal{Y}$, which we denote by γ_ρ.

Remark. Interested readers may find some informative illustrations of $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ and $\mathcal{Y}$ in Scholze's Berkeley lectures [**SW20**, §12].

Definition 1.3.9. Let (R, R^+) be a Huber pair. A *rational subset* of $\mathrm{Spa}(R, R^+)$ is a subset of the form

$$\mathcal{U}(T/g) := \left\{ x \in \mathrm{Spa}(R, R^+) : |f(x)| \leq |g(x)| \neq 0 \text{ for all } f \in T \right\}$$

for some $g \in R$ and some nonempty finite set $T \subseteq R$ such that TR is open in R.

Example 1.3.10. We say that a subset of $\mathcal{Y}$ is *distinguished* if it has the form

$$\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]} := \left\{ x \in \mathcal{Y} : \left|[\varpi^i](x)\right| \leq |p(x)| \leq \left|[\varpi^j](x)\right| \right\}$$

for some $i, j \in \mathbb{Z}[1/p]$ with $0 < j \leq i$. Every distinguished subset of $\mathcal{Y}$ is a rational subset of $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$; indeed, we have an identification

$$\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]} = \left\{ x \in \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) : \left|[\varpi^{i+j}](x)\right|, \left|p^2(x)\right| \leq \left|[\varpi^j]p(x)\right| \neq 0 \right\} = \mathcal{U}(T_{[i,j]}/[\varpi^j]p)$$

where $T_{i,j} := \left\{ [\varpi^{i+j}], p^2 \right\}$ generates an open ideal in A_{inf}. In particular, every distinguished subset of $\mathcal{Y}$ is open in $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$.

Let us describe some points on each $\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}$ in line with our discussion in Example 1.3.8. For an element $y \in Y$, we have $\widetilde{y} \in \mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}$ if and only if y is an element of $Y_{[|\varpi|^i, |\varpi|^j]}$. For a real number ρ with $0 < \rho < 1$, we have $\gamma_\rho \in \mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}$ if and only if ρ belongs to the interval $[|\varpi|^i, |\varpi|^j]$.

Remark. We can extend our discussion above by defining the *absolute value* for an arbitrary point $x \in \mathcal{Y}$. We say that a valuation is of rank 1 if it takes values in the set of positive real numbers. It is a fact that x admits a unique maximal generization $x^{\max}$ of rank 1. We define the absolute value of x by

$$|x| := |\varpi|^{\frac{\log(|p(x^{\max})|)}{\log(|[\varpi](x^{\max})|)}}.$$

Let us now consider $\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}$ of $\mathcal{Y}$ for some $i, j \in \mathbb{Z}[1/p]$. Since $\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}$ is open in $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ as noted above, the point x lies in $\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}$ if and only if $x^{\max}$ does, which amounts to having $|x| \in [|\varpi|^i, |\varpi|^j]$.

Proposition 1.3.11. Let (R, R^+) be a Huber pair, and write $\mathcal{S} := \operatorname{Spa}(R, R^+)$. Consider a rational subset $\mathcal{U} := \mathcal{U}(T/g)$ for some $g \in R$ and some nonempty finite set $T \subseteq R$ such that TR is open in R.

(1) There exists a map of Huber pairs $(R, R^+) \longrightarrow (\mathcal{O}_{\mathcal{S}}(\mathcal{U}), \mathcal{O}_{\mathcal{S}}^+(\mathcal{U}))$ for some complete Huber ring $\mathcal{O}_{\mathcal{S}}(\mathcal{U})$ with the following properties:
   (i) The induced map $\operatorname{Spa}(\mathcal{O}_{\mathcal{S}}(\mathcal{U}), \mathcal{O}_{\mathcal{S}}^+(\mathcal{U})) \longrightarrow \mathcal{S}$ yields a homeomorphism onto $\mathcal{U}$.
   (ii) It is universal for maps of Huber pairs $(R, R^+) \longrightarrow (Q, Q^+)$ such that the induced map $\operatorname{Spa}(Q, Q^+) \longrightarrow \mathcal{S}$ factors over $\mathcal{U}$.

(2) If R is uniform such that the topology on R° is given by a finitely generated ideal I, then $\mathcal{O}_{\mathcal{S}}(\mathcal{U})$ is given by the completion of $R[1/g]$ with respect to the ideal generated by I and the set $T' := \{ f/g : f \in T \}$.

Definition 1.3.12. Let (R, R^+) be a Huber pair, and write $\mathcal{S} := \operatorname{Spa}(R, R^+)$. We define the presheaves $\mathcal{O}_{\mathcal{S}}$ and $\mathcal{O}_{\mathcal{S}}^+$ on $\mathcal{S}$ by

$$\mathcal{O}_{\mathcal{S}}(\mathcal{W}) := \varprojlim_{\substack{\mathcal{U} \subseteq \mathcal{W} \\ \mathcal{U} \text{ rational}}} \mathcal{O}_{\mathcal{S}}(\mathcal{U}) \quad \text{and} \quad \mathcal{O}_{\mathcal{S}}^+(\mathcal{W}) := \varprojlim_{\substack{\mathcal{U} \subseteq \mathcal{W} \\ \mathcal{U} \text{ rational}}} \mathcal{O}_{\mathcal{S}}^+(\mathcal{U}) \quad \text{for all open } \mathcal{W} \subseteq \mathcal{S}$$

where $\mathcal{O}_{\mathcal{S}}(\mathcal{U})$ and $\mathcal{O}_{\mathcal{S}}^+(\mathcal{U})$ for each rational subset $\mathcal{U}$ of $\mathcal{S}$ are given by Proposition 1.3.11. We refer to $\mathcal{O}_{\mathcal{S}}$ as the *structure presheaf* of $\mathcal{S}$.

Remark. The ring $\mathcal{O}_{\mathcal{S}}^+(\mathcal{W})$ is in general not open in $\mathcal{O}_{\mathcal{S}}(\mathcal{W})$.

Example 1.3.13. Let us write $\mathcal{S} := \operatorname{Spa}(A_{\text{inf}}, A_{\text{inf}})$. We assert that $\mathcal{Y}$ is an open subset of $\mathcal{S}$ with $\mathcal{O}_{\mathcal{S}}(\mathcal{Y}) \cong B$. The set $\mathcal{Y}$ is covered by the distinguished subsets; indeed, as both $[\varpi]$ and p are topologically nilpotent in A_{inf}, for every $x \in \mathcal{Y}$ there exist some positive real numbers $i, j \in \mathbb{Z}[1/p]$ with $\left|[\varpi^i](x)\right| \leq |p(x)|$ and $\left|p^{1/j}(x)\right| \leq |[\varpi](x)|$, or equivalently $\left|[\varpi^i](x)\right| \leq |p(x)| \leq \left|[\varpi^j](x)\right|$. Since distinguished subsets of $\mathcal{Y}$ are (open) rational subsets of $\mathcal{S}$ as noted in Example 1.3.10, we deduce that $\mathcal{Y}$ is an open subset of $\mathcal{S}$ with

$$\mathcal{O}_{\mathcal{S}}(\mathcal{Y}) = \varprojlim \mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}) \tag{1.9}$$

where the limit is taken over all distinguished subsets of $\mathcal{Y}$.

Consider arbitrary numbers $i, j \in \mathbb{Z}[1/p]$ with $0 < j \leq i$. In light of (1.9) it suffices to establish an identification

$$\mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}) \cong B_{[|\varpi|^i, |\varpi|^j]}. \tag{1.10}$$

Proposition 1.3.11 and Example 1.3.2 together imply that $\mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]})$ is the completion of $A_{\text{inf}}[1/p, 1/[\varpi]]$ with respect to the ideal I generated by the set $T := \left\{ p, [\varpi], [\varpi^i]/p, p/[\varpi^j] \right\}$. Moreover, the ideal I is generated by $[\varpi^i]/p$ and $p/[\varpi^j]$ as we have $p = (p/[\varpi^j]) \cdot [\varpi^j]$ and $[\varpi] = ([\varpi^i]/p)^r \cdot p^r \cdot (1/[\varpi])^s$ for some positive integers r and s. It is then straightforward to verify that the I-adic topology on $A_{\text{inf}}[1/p, 1/[\varpi]]$ coincides with the topology induced by the Gauss $|\varpi|^i$-norm and the Gauss $|\varpi|^j$-norm. Therefore we obtain the identification (1.10) as desired.

Definition 1.3.14. We say that a Huber pair (R, R^+) is *sheafy* if the structure presheaf on $\operatorname{Spa}(R, R^+)$ is a sheaf.

Proposition 1.3.15. Let (R, R^+) be a Huber pair, and write $\mathcal{S} := \operatorname{Spa}(R, R^+)$.

(1) For every open $\mathcal{W} \subseteq \mathcal{S}$ we have
$$\mathcal{O}_{\mathcal{S}}^+(\mathcal{W}) = \{ f \in \mathcal{O}_{\mathcal{S}}(\mathcal{W}) : |f(x)| \leq 1 \text{ for all } x \in \mathcal{W} \}.$$

(2) The presheaf $\mathcal{O}_{\mathcal{S}}^+$ is a sheaf if (R, R^+) is sheafy.

Definition 1.3.16. Let R be a Huber ring.

(1) We say that R is *Tate* if it contains a topologically nilpotent unit.
(2) We say that R is *strongly noetherian* if for every $n \geq 0$ the Tate algebra
$$R\langle u_1, \cdots, u_n\rangle := \left\{ \sum a_{i_1,\cdots,i_n} u_1^{i_1} \cdots u_n^{i_n} \in R[[u_1, \cdots, u_n]] : \lim a_{i_1,\cdots,i_n} = 0 \right\}$$
is noetherian.

Theorem 1.3.17 (Huber [**Hub94**]). A Huber pair (R, R^+) is sheafy if R is Tate and strongly noetherian.

Theorem 1.3.18 (Kedlaya [**Ked16**]). For every closed interval $[a, b] \subseteq (0, 1)$ the topological ring $B_{[a,b]}$ is a Tate and strongly noetherian Huber ring.

Definition 1.3.19. An *adic space* is a topological space $\mathcal{S}$ together with a sheaf $\mathcal{O}_\mathcal{S}$ of topological rings and a continuous multiplicative valuation v_x on $\mathcal{O}_{\mathcal{S},x}$ for each $x \in \mathcal{S}$ such that $\mathcal{S}$ is locally of the form $\mathrm{Spa}(R, R^+)$ for some sheafy Huber pair (R, R^+).

Example 1.3.20. By Example 1.3.13, Theorem 1.3.17 and Theorem 1.3.18 we deduce that distinguished subsets of $\mathcal{Y}$ are noetherian adic spaces, and in turn find that $\mathcal{Y}$ is a locally noetherian adic space. In addition, for every closed interval $[a, b] \subseteq (0, 1)$ we see that
$$\mathcal{Y}_{[a,b]} := \bigcup_{[|\varpi|^i,|\varpi|^j]\subseteq[a,b]} \mathcal{Y}_{[|\varpi|^i,|\varpi|^j]}$$
is a locally noetherian adic space with $\mathcal{O}_{\mathcal{Y}_{[a,b]}}(\mathcal{Y}_{[a,b]}) = B_{[a,b]}$.

Proposition 1.3.21. Every morphism of Huber pairs $g : (R, R^+) \longrightarrow (Q, Q^+)$ induces a map of presheaves $\mathcal{O}_\mathcal{S} \longrightarrow g_*\mathcal{O}_\mathcal{T}$ where we write $\mathcal{S} := \mathrm{Spa}(R, R^+)$ and $\mathcal{T} := \mathrm{Spa}(Q, Q^+)$.

Example 1.3.22. Let ϕ denote the automorphism of $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ induced by the Frobenius automorphism of A_{inf}. It is evident by construction that $\mathcal{Y}$ is stable under ϕ. In addition, by Example 1.3.13 and Proposition 1.3.21 we get an induced automorphism on $\mathcal{O}_\mathcal{Y}(\mathcal{Y}) \cong B$ which is easily seen to coincide with φ.

Let us choose $c \in (1/p, p) \cap \mathbb{Q}$. For every $n \in \mathbb{Z}$, we set
$$\mathcal{V}_n := \mathcal{Y}_{[|\varpi|^{1/p^n},|\varpi|^{c/p^n}]} \qquad \text{and} \qquad \mathcal{W}_n := \mathcal{Y}_{[|\varpi|^{c/p^n},|\varpi|^{c/p^{n+1}}]}.$$
Arguing as in Example 1.3.13, we find that $\mathcal{Y}$ is covered by such sets. In addition, we have $\phi(\mathcal{V}_n) = \mathcal{V}_{n-1}$ and $\phi(\mathcal{W}_n) = \mathcal{W}_{n-1}$ for all $n \in \mathbb{Z}$. Therefore the action of ϕ on $\mathcal{Y}$ is properly discontinuous, and consequently yields the quotient space
$$\mathcal{X} := \mathcal{Y}/\phi^\mathbb{Z}.$$
Moreover, $\mathcal{X}$ is covered by (the isomorphic images of) $\mathcal{V}_0$ and $\mathcal{W}_0$, which are noetherian adic spaces as noted in Example 1.3.20. Hence $\mathcal{X}$ is a noetherian adic space with $\mathcal{O}_\mathcal{X}(\mathcal{X}) \cong B^{\varphi=1}$.

Definition 1.3.23. We refer to the noetherian adic space $\mathcal{X}$ constructed in Example 1.3.22 as the *adic Fargues-Fontaine curve*.

Theorem 1.3.24 (Kedlaya-Liu [**KL15**]). There exists a natural morphism of locally ringed spaces $h : \mathcal{X} \longrightarrow X$ such that the pullback along h induces an equivalence
$$h^* : \mathrm{Bun}_X \xrightarrow{\sim} \mathrm{Bun}_\mathcal{X}$$
where Bun_X and $\mathrm{Bun}_\mathcal{X}$ respectively denote the categories of vector bundles on X and $\mathcal{X}$.

Remark. Theorem 1.3.24 is often referred to as "GAGA for the Fargues-Fontaine curve". By Theorem 1.3.24, studying the schematic Fargues-Fontaine curve is essentially equivalent to studying the adic Fargues-Fontaine curve.

2. Geometric structure

In this section we establish some fundamental geometric properties of the Fargues-Fontaine curve. Our discussion will show that the Fargues-Fontaine curve is geometrically very akin to proper curves over $\mathbb{Q}_p$. In addition, our discussion will provide a number of new perspectives towards several constructions from Chapter III. The primary references for this section are Fargues and Fontaine's survey paper [**FF12**] and Lurie's notes [**Lur**]

2.1. Legendre-Newton polygons

We begin by introducing a crucial tool for studying the structure of the ring B.

Definition 2.1.1. Let $\log_p$ denote the real logarithm base p.

(1) Given an element $f \in B$, we define the *Legendre-Newton polygon* of f as the function $\mathcal{L}_f : (0, \infty) \longrightarrow \mathbb{R} \cup \{ \infty \}$ given by

$$\mathcal{L}_f(s) := -\log_p \left(|f|_{p^{-s}} \right) \qquad \text{for all } s \in (0, \infty).$$

(2) Given a closed interval $[a, b] \subseteq (0, 1)$ and an element $f \in B_{[a,b]}$, we define the *Legendre-Newton* $[a, b]$*-polygon* of f as the function $\mathcal{L}_{f,[a,b]} : [-\log_p(b), -\log_p(a)] \longrightarrow \mathbb{R} \cup \{ \infty \}$ given by

$$\mathcal{L}_{f,[a,b]}(s) := -\log_p \left(|f|_{p^{-s}} \right) \qquad \text{for all } s \in [-\log_p(b), -\log_p(a)].$$

Remark. With notations as in Example 1.3.8, we may write $\mathcal{L}_f(s) = -\log_p \left(\left| f(\gamma_{p^{-s}}) \right| \right)$ for all $f \in B$ and $s \in (0, \infty)$.

LEMMA 2.1.2. Given any elements $f, g \in A_{\text{inf}}[1/p, 1/[\varpi]]$, we have

$$\mathcal{L}_{fg}(s) = \mathcal{L}_f(s) + \mathcal{L}_g(s) \quad \text{and} \quad \mathcal{L}_{f+g}(s) \geq \min(\mathcal{L}_f(s), \mathcal{L}_g(s)) \quad \text{for all } s \in (0, \infty).$$

PROOF. This is an immediate consequence of Proposition 1.2.9. □

Our main goal in this subsection is to prove that Legendre-Newton polygons are indeed polygons with decreasing integer slopes.

Definition 2.1.3. Let g be a piecewise linear function defined on an interval $I \subseteq \mathbb{R}$.

(1) We say that g is *concave* if the slopes are decreasing, and *convex* if the slopes are increasing.
(2) We write $\partial_- g$ and $\partial_+ g$ respectively for the left and right derivatives of g.

Example 2.1.4. Let $f = \sum [c_n] p^n$ be a nonzero element in $A_{\text{inf}}[1/p, 1/[\varpi]]$. Its *Newton polygon* is defined as the lower convex hull the points $(n, \nu_F(c_n)) \in \mathbb{R}^2$, which we may regard as a convex piecewise linear function on $(0, \infty)$.

LEMMA 2.1.5. Given a nonzero element $f = \sum [c_n] p^n \in A_{\text{inf}}[1/p, 1/[\varpi]]$, we have

$$\mathcal{L}_f(s) = \inf_{n \in \mathbb{Z}} (\nu_F(c_n) + ns) \qquad \text{for every } s \in (0, \infty).$$

PROOF. This is obvious by definition. □

Remark. By Lemma 2.1.5 it is not hard to verify that $\mathcal{L}_f$ coincides with the Legendre transform of the Newton polygon of f.

Example 2.1.6. Let ξ be a primitive element in A_{inf} with the Teichmüller expansion $\xi = \sum [c_n] p^n$. By Proposition 1.1.12 we have

$$\mathcal{L}_\xi(s) = \min(\nu_F(c_0), \nu_F(c_1) + s) = \min(\nu_F(c_0), s) \qquad \text{for all } s \in (0, \infty).$$

PROPOSITION 2.1.7. Let $f = \sum [c_n]$ be a nonzero element in $A_{\text{inf}}[1/p, 1/[\varpi]]$.

(1) $\mathcal{L}_f$ is a concave piecewise linear function with integer slopes.
(2) For each $s \in (0, \infty)$, the one-sided derivatives $\partial_- \mathcal{L}_f(s)$ and $\partial_+ \mathcal{L}_f(s)$ are respectively given by the minimum and maximum elements of the set

$$T_s := \{ n \in \mathbb{Z} : \mathcal{L}_f(s) = \nu_F(c_n) + ns \}.$$

PROOF. Fix a real number $s > 0$. Lemma 2.1.5 and Lemma 1.2.5 together imply that T_s is finite. Let l and r respectively denote the minimum and maximum elements of T_s. By construction we have

$$\nu_F(c_l) + ls = \nu_F(c_r) + rs \leq \nu_F(c_n) + ns \qquad \text{for all } n \in \mathbb{Z} \tag{2.1}$$

where equality holds if and only if n belongs to T_s. It suffices to show that for all sufficiently small $\epsilon > 0$ we have

$$\mathcal{L}_f(s + \epsilon) = \mathcal{L}_f(s) + l\epsilon \qquad \text{and} \qquad \mathcal{L}_f(s - \epsilon) = \mathcal{L}_f(s) - r\epsilon. \tag{2.2}$$

Let us consider the first identity in (2.2). Take $k < 0$ with $c_n = 0$ for all $n \leq k$, and set

$$\delta_1 := \inf_{n<l} \left(\frac{(\nu_F(c_n) + ns) - (\nu_F(c_l) + ls)}{l - n} \right) = \inf_{k<n<l} \left(\frac{(\nu_F(c_n) + ns) - (\nu_F(c_l) + ls)}{l - n} \right).$$

Then we have $\delta_1 > 0$ as the inequality in (2.1) is strict for all $n < l$. Let ϵ be a real number with $0 < \epsilon < \delta_1$. For every $n < l$ we find $\epsilon(l - n) < \delta_1(l - n) \leq (\nu_F(c_n) + ns) - (\nu_F(c_l) + ls)$ and consequently obtain

$$\nu_F(c_l) + l(s + \epsilon) < \nu_F(c_n) + n(s + \epsilon).$$

In addition, for every $n > l$ we have

$$\nu_F(c_l) + l(s + \epsilon) \leq \nu_F(c_n) + ns + l\epsilon < \nu_F(c_n) + n(s + \epsilon)$$

where the first inequality follows from (2.1). Therefore we obtain

$$\mathcal{L}_f(s + \epsilon) = \inf_{n \in \mathbb{Z}} (\nu_F(c_n) + n(s + \epsilon)) = \nu_F(c_l) + l(s + \epsilon) = \mathcal{L}_f(s) + l\epsilon.$$

We now consider the second identity in (2.2). Proposition 1.2.3 implies that there exists $\lambda \in \mathbb{R}$ with $\nu_F(c_n) > \lambda$ for all $n \in \mathbb{Z}$. Let us set

$$u := \frac{\nu_F(c_r) - \lambda}{s/2} + r \quad \text{and} \quad \delta_2 := \inf_{r<n<u} \left(\frac{(\nu_F(c_n) + ns) - (\nu_F(c_r) + rs)}{n - r} \right).$$

Then we have $\delta_2 > 0$ as the inequality in (2.1) is strict for all $n > r$. Let ϵ be a real number with $0 < \epsilon < \min(s/2, \delta_2)$. For every $n > u$ we find

$$\nu_F(c_r) - \nu_F(c_n) < \nu_F(c_r) - \lambda = (u - r)s/2 < (n - r)(s - \epsilon)$$

and consequently obtain

$$\nu_F(c_r) + r(s - \epsilon) < \nu_F(c_n) + n(s - \epsilon).$$

In addition, we get the same inequality for every $n < r$ by arguing as in the preceding paragraph. Therefore we deduce

$$\mathcal{L}_f(s - \epsilon) = \inf_{n \in \mathbb{Z}} (\nu_F(c_n) + n(s - \epsilon)) = \nu_F(c_r) + r(s - \epsilon) = \mathcal{L}_f(s) - r\epsilon,$$

thereby completing the proof. □

Remark. In light of the remark after Lemma 2.1.5, we can alternatively deduce Proposition 2.1.7 from a general fact that the Legendre transform of a convex piecewise linear function with integer breakpoints is a concave piecewise linear function with integer slopes.

LEMMA 2.1.8. Let (f_n) be a Cauchy sequence in $A_{\inf}[1/p, 1/[\varpi]]$ with respect to the Gauss p^{-s}-norm for some $s > 0$. Assume that (f_n) does not converge to 0. Then the sequences $(\mathcal{L}_{f_n}(s))$, $(\partial_-\mathcal{L}_{f_n}(s))$, and $(\partial_+\mathcal{L}_{f_n}(s))$ are all eventually constant.

PROOF. The sequence $\left(|f_n|_{p^{-s}}\right)$ converges in $\mathbb{R}$. Let us set

$$a := \lim_{n\to\infty} \mathcal{L}_{f_n}(s) = -\lim_{n\to\infty} \log_p \left(|f_n|_{p^{-s}}\right),$$

and take an integer $u > 0$ with

$$\mathcal{L}_{f_n - f_u}(s) = -\log_p \left(|f_n - f_u|_{p^{-s}}\right) > 2a \qquad \text{and} \qquad \mathcal{L}_{f_n}(s) < 2a \qquad \text{for all } n \geq u.$$

For every $n \geq u$, since both $\mathcal{L}_{f_u}$ and $\mathcal{L}_{f_n - f_u}$ are continuous, we may find some $\delta_n > 0$ with

$$\mathcal{L}_{f_n - f_u}(s + \epsilon) > 2a > \mathcal{L}_{f_u}(s + \epsilon) \qquad \text{for all } \epsilon \in (-\delta_n, \delta_n),$$

and consequently obtain $\mathcal{L}_{f_u}(s + \epsilon) = \mathcal{L}_{f_n}(s + \epsilon)$ for all $\epsilon \in (-\delta_n, \delta_n)$ by Lemma 2.1.2. This implies that for every $n \geq u$ we have

$$\mathcal{L}_{f_n}(s) = \mathcal{L}_{f_u}(s), \qquad \partial_-\mathcal{L}_{f_n}(s) = \partial_-\mathcal{L}_{f_u}(s), \qquad \partial_+\mathcal{L}_{f_n}(s) = \partial_+\mathcal{L}_{f_u}(s).$$

Hence we deduce the desired assertion. □

PROPOSITION 2.1.9. Let $[a, b]$ be a closed subinterval of $(0, 1)$, and let (f_n) be a Cauchy sequence in $A_{\inf}[1/p, 1/[\varpi]]$ with respect to the Gauss a-norm and the Gauss b-norm. Assume that (f_n) does not converge to 0 with respect to either the Gauss a-norm or the Gauss b-norm. Then the sequence of functions $(\mathcal{L}_{f_n})$ is eventually constant on $[-\log_p(b), -\log_p(a)]$.

PROOF. Let us write $l := -\log_p(b)$ and $r := -\log_p(a)$. Without loss of generality we may assume that each f_n is not zero. In addition, by symmetry we may assume that f_n does not converge to 0 with respect to the Gauss b-norm. Then Lemma 2.1.8 yields $\alpha, \beta \in \mathbb{R}$ and $u \in \mathbb{Z}$ such that we have $\mathcal{L}_{f_n}(l) = \alpha$ and $\partial_+\mathcal{L}_{f_n}(l) = \beta$ for all $n > u$. Since each $\mathcal{L}_{f_n}$ is concave and piecewise linear by Proposition 2.1.7, we set $\omega := \max(\alpha, \alpha + \beta(r - l))$ and find

$$\mathcal{L}_{f_n}(s) \leq \alpha + \beta(s - l) \leq \omega \qquad \text{for all } n > u \text{ and } s \in [l, r]. \tag{2.3}$$

Moreover, Lemma 1.2.11 (or Proposition 1.2.12) implies that the sequence (f_n) converges with respect to all Gauss ρ-norms with $\rho \in [a, b]$, thereby yielding an integer $u' > u$ with $|f_n - f_{u'}|_\rho < p^{-\omega}$ for all $n > u'$ and $\rho \in [a, b]$, or equivalently

$$\mathcal{L}_{f_n - f_{u'}}(s) > \omega \qquad \text{for all } n > u' \text{ and } s \in [l, r].$$

Hence by Lemma 2.1.2 and (2.3) we find

$$\mathcal{L}_{f_n}(s) = \mathcal{L}_{f_{u'}}(s) \qquad \text{for all } n > u' \text{ and } s \in [l, r].$$

thereby deducing the desired assertion. □

PROPOSITION 2.1.10. Let $[a, b]$ be a closed subinterval of $(0, 1)$. For every nonzero $f \in B_{[a,b]}$, the function $\mathcal{L}_{f,[a,b]}$ is concave and piecewise linear with integer slopes.

PROOF. Take a sequence (f_n) in $A_{\inf}[1/p, 1/[\varpi]]$ which converges to f with respect to the Gauss a-norm and the Gauss b-norm. By Proposition 1.2.12 we have

$$\mathcal{L}_{f,[a,b]}(s) = \lim_{n\to\infty} \mathcal{L}_{f_n}(s) \qquad \text{for all } s \in [-\log_p(b), -\log_p(a)].$$

Since f is not zero, the assertion follows by Proposition 2.1.7 an Proposition 2.1.9. □

Remark. For a holomorphic function g on the annulus $\mathbb{D}^*_{[a,b]} := \{ z \in \mathbb{C} : a \leq |z| \leq b \}$, the Hadamard three-circle theorem asserts that the function $\mathcal{M}_g : [\ln(a), \ln(b)] \longrightarrow \mathbb{R}$ defined by $\mathcal{M}_g(r) := \ln\left(\sup_{|z|=e^r} (|g(z)|) \right)$ for all $r \in [\ln(a), \ln(b)]$ is convex. In light of the remark after Lemma 1.2.11 we may consider Proposition 2.1.10 as an analogue of the Hadamard three-circle theorem.

Corollary 2.1.11. For every nonzero $f \in B$, the Legendre-Newton polygon $\mathcal{L}_f$ is a concave piecewise linear function with integer slopes.

Remark. Corollary 2.1.11 suggests that we can define the Newton polygon of f as the Legendre transform of $\mathcal{L}_f$.

Example 2.1.12. Let f be an invertible element in B. By Lemma 2.1.2 we find

$$\mathcal{L}_f(s) = \mathcal{L}_1(s) - \mathcal{L}_{f^{-1}}(s) = -\mathcal{L}_{f^{-1}}(s) \qquad \text{for all } s \in (0, \infty).$$

Since both $\mathcal{L}_f$ and $\mathcal{L}_{f^{-1}}$ are concave piecewise linear functions as noted in Corollary 2.1.11, we deduce that $\mathcal{L}_f$ is linear.

Remark. In fact, it is not hard to prove that a nonzero element $f \in B$ is invertible if and only if $\mathcal{L}_f$ is linear.

Let us present some important applications of the Legendre-Newton polygons.

Definition 2.1.13. For every $n \in \mathbb{Z}$, we refer to the ring $B^{\varphi=p^n}$ as the *Frobenius eigenspace* of B with eigenvalue p^n.

Lemma 2.1.14. Given an element $f \in B$, we have

$$|\varphi(f)|_{\rho^p} = |f|_\rho^p \qquad \text{and} \qquad |pf|_\rho = \rho\,|f|_\rho \qquad \text{for all } \rho \in (0, 1).$$

Proof. If f is an element in $A_{\inf}[1/p, 1/[\varpi]]$, the assertion is evident by construction. The assertion for the general case then follows by continuity. □

Proposition 2.1.15. The Frobenius eigenspace $B^{\varphi=p^n}$ is trivial for every $n < 0$.

Proof. Suppose for contradiction that $B^{\varphi=p^n}$ contains a nonzero element f. By Lemma 2.1.14 we have

$$p\mathcal{L}_f(s) = \mathcal{L}_{\varphi(f)}(ps) = \mathcal{L}_{p^n f}(ps) = nps + \mathcal{L}_f(ps) \qquad \text{for all } s > 0.$$

Since $\mathcal{L}_f$ is a concave piecewise linear function by Corollary 2.1.11, we find

$$p\partial_+\mathcal{L}_f(s) = np + p\partial_+\mathcal{L}_f(ps) \leq np + p\partial_+\mathcal{L}_f(s) \qquad \text{for all } s > 0, \tag{2.4}$$

thereby obtaining a contradiction as desired. □

Remark. A similar argument shows that $\mathcal{L}_f$ is linear for every nonzero $f \in B^{\varphi=1}$. In Proposition 3.1.6 we will build on this fact to prove that $B^{\varphi=1}$ is naturally isomorphic to $\mathbb{Q}_p$.

Proposition 2.1.16. Let $[a, b]$ be a closed subinterval of $(0, 1)$, and let f be a nonzero element in $B_{[a,b]}$. Then we have $|f|_\rho \neq 0$ for every $\rho \in [a, b]$.

Proof. Proposition 2.1.10 implies that $\mathcal{L}_{f,[a,b]}(-\log_p(\rho)) = -\log_p\left(|f|_\rho\right)$ is finite for every $\rho \in [a, b]$, thereby yielding the desired assertion. □

Corollary 2.1.17. For every closed interval $[a, b] \subseteq (0, 1)$ the ring $B_{[a,b]}$ is an integral domain.

Proof. This is an immediate consequence of Proposition 1.2.9 and Proposition 2.1.16. □

2.2. Divisors and zeros of functions

In this subsection we define the notion of divisors on Y for elements in B.

Definition 2.2.1. A *divisor* on Y is a formal sum $\sum_{y \in Y} n_y \cdot y$ with $n_y \in \mathbb{Z}$ such that for every closed interval $[a,b] \subseteq (0,1)$ the set $Z_{[a,b]} := \left\{ y \in Y_{[a,b]} : n_y \neq 0 \right\}$ is finite.

Remark. Definition 2.2.1 is comparable with the definition of Weil divisors on locally noetherian integral schemes as given in [**Sta**, Tag 0BE2].

LEMMA 2.2.2. Let f and g be elements in B. Assume that f is divisible by g in $B_{[a,b]}$ for every closed interval $[a,b] \subseteq (0,1)$. Then f is divisible by g in B.

PROOF. For every $n \geq 2$ we may write $f = gh_n$ for some $h_n \in B_{[1/n, 1-1/n]}$. Then by Corollary 1.2.13 and Corollary 2.1.17 we find that h_n takes a constant value for all $n \geq 2$. Hence we get an element $h \in B$ with $h = h_n$ for all $n \geq 2$, thereby obtaining the desired assertion. ☐

PROPOSITION 2.2.3. Let y be an element in Y, represented by an untilt C of F. Every $f \in B$ with $f(y) = 0$ is divisible by every primitive element $\xi \in \ker(\theta_C)$.

PROOF. Consider an arbitrary closed interval $[a,b] \subseteq (0,1)$ with $y \in Y_{[a,b]}$. By Lemma 2.2.2 it suffices to prove that f is divisible by ξ in $B_{[a,b]}$. Take a sequence (f_n) in $A_{\text{inf}}[1/p, 1/[\varpi]]$ which converges to f with respect to the Gauss a-norm and the Gauss b-norm. By Corollary 1.1.7 we may write $f_n(y) = c_n^\sharp$ for some $c_n \in F$. Then we have

$$\lim_{n\to\infty} |c_n| = \lim_{n\to\infty} \left|c_n^\sharp\right|_C = \lim_{n\to\infty} |f_n(y)|_C = |f(y)|_C = 0,$$

and consequently find that the sequence $([c_n])$ converges to 0 with respect to the Gauss a-norm and the Gauss b-norm. Hence we may replace (f_n) by $(f_n - [c_n])$ to assume $f_n(y) = 0$ for all $n > 0$.

Let $\widetilde{\theta_C} : A_{\text{inf}}[1/p, 1/[\varpi]] \longrightarrow C$ be the ring homomorphism which extends the untilt map θ_C. Proposition 1.1.19 implies that ξ generates $\ker(\widetilde{\theta_C})$. We may thus write $f_n = \xi g_n$ for some $g_n \in A_{\text{inf}}[1/p, 1/[\varpi]]$. Then for every $\rho \in [a,b]$ we use Proposition 1.2.9 to find

$$\lim_{n\to\infty} |g_{n+1} - g_n|_\rho = \frac{1}{|\xi|_\rho} \cdot \lim_{n\to\infty} |\xi(g_{n+1} - g_n)|_\rho = \frac{1}{|\xi|_\rho} \cdot \lim_{n\to\infty} |f_{n+1} - f_n|_\rho = 0,$$

which means that the sequence (g_n) is Cauchy with respect to the Gauss ρ-norm. Therefore the sequence (g_n) defines an element $g \in B_{[a,b]}$ with $f = \xi g$. ☐

Remark. By Corollary 1.1.7 we may write $p = (p^\flat)^\sharp$ for some $p^\flat \in \mathfrak{m}_F$, which is uniquely determined up to unit multiple. Then we obtain a primitive element $[p^\flat] - p \in \ker(\theta_C)$, and consequently find an expression $f = ([p^\flat] - p)g$ for some $g \in B$ by Proposition 2.2.3. This is an analogue of the fact that a holomorphic function f on $\mathbb{D}^*$ with a zero at $z_0 \in \mathbb{D}^*$ can be written in the form $f = (z - z_0)g$ for some holomorphic function g on $\mathbb{D}^*$.

COROLLARY 2.2.4. Let C be a characteristic 0 untilt of F. Every primitive element $\xi \in \ker(\theta_C)$ generates $\ker(\widehat{\theta_C})$.

Remark. Let $[a,b]$ be a closed subinterval of $(0,1)$ with $|p|_C \in [a,b]$. By the proof of Proposition 1.2.16 the untilt map θ_C extends to a surjective continuous ring homomorphism $\widehat{\widehat{\theta_C}} : B_{[a,b]} \twoheadrightarrow C$. Then we can similarly show that every primitive element $\xi \in \ker(\theta_C)$ generates $\ker(\widehat{\widehat{\theta_C}})$.

PROPOSITION 2.2.5. Let C be a characteristic 0 untilt of F, and let $\theta_C[1/p] : A_{\text{inf}}[1/p] \longrightarrow C$ be the ring homomorphism which extends the untilt map θ_C. Then we have

$$A_{\text{inf}}[1/p] \cap \ker(\widehat{\theta_C})^j = \ker(\theta_C[1/p])^j \qquad \text{for all } j \geq 1.$$

PROOF. The assertion for $j = 1$ follows by observing that $\widehat{\theta_C}$ restricts to $\theta_C[1/p]$. Let us now proceed by induction on j. We only need to show $A_{\text{inf}}[1/p] \cap \ker(\widehat{\theta_C})^j \subseteq \ker(\theta_C[1/p])^j$, since the reverse containment is obvious by the fact that $\widehat{\theta_C}$ restricts to $\theta_C[1/p]$. Let a be an arbitrary element in $A_{\text{inf}}[1/p] \cap \ker(\widehat{\theta_C})^j$, and choose a primitive element $\xi \in \ker(\theta_C)$. Then ξ generates both $\ker(\widehat{\theta_C})$ and $\ker(\theta_C[1/p])$ by Corollary 2.2.4 and Proposition 1.1.19. Hence we may write $a = \xi^j b$ for some $b \in B$. In addition, since we have

$$A_{\text{inf}}[1/p] \cap \ker(\widehat{\theta_C})^j \subseteq A_{\text{inf}}[1/p] \cap \ker(\widehat{\theta_C})^{j-1} = \ker(\theta_C[1/p])^{j-1}$$

by the induction hypothesis, there exists some $c \in A_{\text{inf}}[1/p]$ with $a = \xi^{j-1}c$. We then find

$$0 = a - a = \xi^j b - \xi^{j-1}c = \xi^{j-1}(\xi b - c),$$

and consequently obtain $c = \xi b$ by Corollary 2.1.17. This implies $c \in A_{\text{inf}}[1/p] \cap \ker(\widehat{\theta_C})$, and in turn yields $c \in \ker(\theta_C[1/p])$ by the assertion for $j = 1$ that we have already established. Therefore we deduce $a = \xi^{j-1}c \in \ker(\theta_C[1/p])^j$ as desired. □

Definition 2.2.6. Let y be an element in Y, represented by an untilt C of F. We define the *de Rham local ring at y* by

$$B_{\text{dR}}^+(y) := \varprojlim_j A_{\text{inf}}[1/p]/\ker(\theta_C[1/p])^j$$

where $\theta_C[1/p] : A_{\text{inf}}[1/p] \longrightarrow C$ is the ring homomorphism which extends the untilt map θ_C.

PROPOSITION 2.2.7. Let y be an element in Y, represented by an untilt C of F.

(1) The ring $B_{\text{dR}}^+(y)$ is a complete discrete valuation ring with C as the residue field.
(2) Every primitive element in $\ker(\theta_C)$ is a uniformizer of $B_{\text{dR}}^+(y)$.
(3) There exists a natural isomorphism

$$B_{\text{dR}}^+(y) \cong \varprojlim_j B/\ker(\widehat{\theta_C})^j.$$

PROOF. Since C is algebraically closed as noted in Proposition 1.1.6, all results from the first part of §2.2 in Chapter III remain valid with C in place of $\mathbb{C}_K$. Hence the statements (1) and (2) follow from Proposition 2.2.16 in Chapter III and Proposition 1.1.19.

It remains to verify the statement (3). Let $\theta_C[1/p] : A_{\text{inf}}[1/p] \twoheadrightarrow C$ be the surjective ring homomorphism which extends the untilt map θ_C, and choose a primitive element $\xi \in \ker(\theta_C)$. Then ξ generates both $\ker(\widehat{\theta_C})$ and $\ker(\theta_C[1/p])$ by Corollary 2.2.4 and Proposition 1.1.19. Hence we get a natural map

$$B_{\text{dR}}^+(y) = \varprojlim_j A_{\text{inf}}[1/p]/\xi^j A_{\text{inf}}[1/p] \longrightarrow \varprojlim_j B/\xi^j B = \varprojlim_j B/\ker(\widehat{\theta_C})^j \tag{2.5}$$

which is easily seen to be injective by Proposition 2.2.5. Moreover, since we have

$$A_{\text{inf}}[1/p]/\xi A_{\text{inf}}[1/p] \cong C \cong B/\xi B,$$

the map (2.5) is surjective by a general fact as stated in [**Sta**, Tag 0315]. We thus deduce that the natural map (2.5) is an isomorphism, thereby completing the proof. □

Definition 2.2.8. Let f be a nonzero element in B. We define its *order of vanishing* at $y \in Y$ to be its valuation in $B_{\mathrm{dR}}^{+}(y)$, denoted by $\mathrm{ord}_y(f)$.

Remark. The element y gives rise to a point $\widetilde{y} \in \mathcal{Y}$ as described in Example 1.3.8. With Proposition 2.2.7 and our discussion in §1.3 we can show that $B_{\mathrm{dR}}^{+}(y)$ is the completed local ring at $\widetilde{y}$. In this sense, Definition 2.2.8 agrees with the usual definition for order of vanishing.

Example 2.2.9. Let ξ be a nondegenerate primitive element in A_{inf}. Theorem 1.1.21 implies that ξ vanishes at a unique element $y_\xi \in Y$. Then we have

$$\mathrm{ord}_y(\xi) = \begin{cases} 1 & \text{for } y = y_\xi, \\ 0 & \text{for } y \neq y_\xi. \end{cases}$$

Lemma 2.2.10. Let f and g be nonzero elements in B. Then we have

$$\mathrm{ord}_y(fg) = \mathrm{ord}_y(f) + \mathrm{ord}_y(g) \qquad \text{for all } y \in Y.$$

Proof. This is evident by definition. □

Proposition 2.2.11. Let f be a nonzero element in B. For every closed interval $[a, b] \subseteq (0, 1)$, the set $Z_{[a,b]} := \left\{ y \in Y_{[a,b]} : \mathrm{ord}_y(f) \neq 0 \right\}$ is finite.

Proof. Let us write $l := -\log_p(b)$ and $r := -\log_p(a)$. We also set $n := \partial_- \mathcal{L}_f(l) - \partial_+ \mathcal{L}_f(r)$, which is a nonnegative integer by Corollary 2.1.11. Since we have $\mathrm{ord}_y(f) \geq 0$ for all $y \in Y$, it suffices to show

$$\sum_{y \in Z_{[a,b]}} \mathrm{ord}_y(f) \leq n. \tag{2.6}$$

Suppose for contradiction that this inequality fails. By Proposition 2.2.3, Example 2.2.9 and Lemma 2.2.10 we may write

$$f = \xi_1 \xi_2 \cdots \xi_{n+1} g \tag{2.7}$$

for some $g \in B$ and primitive elements $\xi_1, \cdots, \xi_{n+1} \in A_{\mathrm{inf}}$ such that each ξ_i vanishes at a unique element $y_i \in Y_{[a,b]}$. Then Example 1.2.2 and Example 2.1.6 together imply that for each $i = 1, \cdots, n+1$ we have

$$\mathcal{L}_{\xi_i}(s) = \begin{cases} s & \text{for } s \leq -\log_p(|y_i|), \\ -\log_p(|y_i|) & \text{for } s > -\log_p(|y_i|). \end{cases}$$

Hence we obtain

$$\partial_- \mathcal{L}_{\xi_i}(l) - \partial_+ \mathcal{L}_{\xi_i}(r) = 1 - 0 = 1 \qquad \text{for each } i = 1, \cdots, n+1.$$

In addition, by Corollary 2.1.11 we have $\partial_- \mathcal{L}_f(l) - \partial_+ \mathcal{L}_f(r) \geq 0$. Therefore we use Lemma 2.1.2 and (2.7) to find

$$\begin{aligned} n &= \partial_- \mathcal{L}_f(l) - \partial_+ \mathcal{L}_f(r) \\ &= \sum_{i=1}^{n+1} \left(\partial_- \mathcal{L}_{\xi_i}(l) - \partial_+ \mathcal{L}_{\xi_i}(r) \right) + \left(\partial_- \mathcal{L}_g(l) - \partial_+ \mathcal{L}_g(r) \right) \\ &\geq n + 1, \end{aligned}$$

thereby obtaining a contradiction as desired. □

Remark. It turns out that the inequality (2.6) is indeed an equality.

Definition 2.2.12. For every $f \in B$, we define its *associated divisor* on Y by

$$\mathrm{Div}(f) := \sum_{y \in Y} \mathrm{ord}_y(f) \cdot y.$$

2.3. The logarithm and untilts

In this subsection, we define and study the logarithms of elements in the multiplicative group $1+\mathfrak{m}_F$. For the rest of this section we write $\mathfrak{m}_F^* := \mathfrak{m}_F \setminus \{0\}$.

PROPOSITION 2.3.1. There exists a group homomorphism $\log : 1+\mathfrak{m}_F \longrightarrow B^{\varphi=p}$ with

$$\log(\varepsilon) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} \qquad \text{for every } \varepsilon \in 1+\mathfrak{m}_F. \tag{2.8}$$

PROOF. Given arbitrary $\varepsilon \in 1+\mathfrak{m}_F$ and $\rho \in (0,1)$, we write $[\varepsilon]-1 = \sum [c_n]p^n$ with $c_n \in \mathcal{O}_F$ to find

$$|[\varepsilon]-1|_\rho \leq \max(|c_0|, \rho) = \max(|\varepsilon-1|, \rho) < 1.$$

Hence we obtain a map $\log : 1+\mathfrak{m}_F \longrightarrow B$ satisfying (2.8). It then follows that log is a group homomorphism by the identity of formal power series $\log(xy) = \log(x)+\log(y)$. Furthermore, as φ is continuous by construction, for every $\varepsilon \in 1+\mathfrak{m}_F$ we find

$$\varphi(\log(\varepsilon)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\varphi([\varepsilon])-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon^p]-1)^n}{n} = \log(\varepsilon^p) = p\log(\varepsilon),$$

thereby completing the proof. □

Remark. We will see in Proposition 3.1.8 that log is a $\mathbb{Q}_p$-linear isomorphism.

Definition 2.3.2. We refer to the map $\log : 1+\mathfrak{m}_F \longrightarrow B^{\varphi=p}$ constructed in Proposition 2.3.1 as the *logarithm* on $1+\mathfrak{m}_F$.

PROPOSITION 2.3.3. Let C be a characteristic 0 untilt of F, and let $\mathfrak{m}_C$ denote the maximal ideal of $\mathcal{O}_C$. There exists a commutative diagram

$$\begin{array}{ccc} 1+\mathfrak{m}_F & \xrightarrow{\log} & B^{\varphi=p} \\ {\scriptstyle \varepsilon\mapsto\varepsilon^\sharp}\downarrow & & \downarrow{\scriptstyle \widehat{\theta_C}} \\ 1+\mathfrak{m}_C & \xrightarrow[\log_{\mu_{p^\infty}}]{} & C \end{array} \tag{2.9}$$

where all maps are group homomorphisms.

PROOF. Let c be an arbitrary element in $\mathcal{O}_F$. By Proposition 2.1.9 in Chapter III, there exists some $a \in \mathcal{O}_C$ with $c^\sharp - 1 = (c-1)^\sharp + pa$. If c belongs to $1+\mathfrak{m}_F$, then we have

$$\left|c^\sharp - 1\right|_C \leq \max\left(\left|(c-1)^\sharp\right|_C, |pa|_C\right) = \max(|c-1|, |pa|_C) < 1$$

and in turn obtain $c^\sharp \in 1+\mathfrak{m}_C$. Conversely, if $c^\sharp$ belongs to $1+\mathfrak{m}_C$, then we have

$$|c-1| = \left|(c-1)^\sharp\right|_C \leq \max\left(\left|c^\sharp - 1\right|_C, pa\right) < 1$$

and consequently obtain $c \in 1+\mathfrak{m}_F$. Therefore in light of Corollary 1.1.7 we deduce that $1+\mathfrak{m}_F$ maps onto $1+\mathfrak{m}_C$ under the sharp map.

Since the map $\widehat{\theta_C}$ is continuous by construction, for every $\varepsilon \in 1+\mathfrak{m}_F$ we have

$$\widehat{\theta_C}(\log(\varepsilon)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\widehat{\theta_C}([\varepsilon])-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\varepsilon^\sharp - 1)^n}{n} = \log_{\mu_{p^\infty}}(\varepsilon^\sharp)$$

where the last identity follows by Example 3.3.9 in Chapter II. Moreover, as C is algebraically closed by Proposition 1.1.6, the map $\log_{\mu_{p^\infty}}$ is a surjective homomorphism by Proposition 3.3.11 in Chapter II. Therefore we obtain the commutative diagram (2.9) as desired. □

PROPOSITION 2.3.4. For every $\varepsilon \in 1 + \mathfrak{m}_F^*$, the element

$$\xi_\varepsilon := \frac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1} = 1 + [\varepsilon^{1/p}] + \cdots + [\varepsilon^{(p-1)/p}] \in A_{\text{inf}}$$

is a nondegenerate primitive element which divides $[\varepsilon] - 1$ but not $[\varepsilon^{1/p}] - 1$.

PROOF. Let us write $k := \mathcal{O}_F/\mathfrak{m}_F$ for the residue field of F, and $W(k)$ for the ring of Witt vectors over k. In addition, for every $c \in \mathcal{O}_F$ we denote by $\overline{c}$ its image under the natural map $\mathcal{O}_F \twoheadrightarrow k$, and by $[\overline{c}]$ the Teichmüller lift of $\overline{c}$ in $W(k)$. Lemma 2.3.1 from Chapter II yields a homomorphism $\pi : A_{\text{inf}} \longrightarrow W(k)$ with

$$\pi\left(\sum [c_n]p^n\right) = \sum [\overline{c_n}]p^n \qquad \text{for all } c_n \in \mathcal{O}_F.$$

We then find $\pi(\xi_\varepsilon) = p$ by observing $\overline{\varepsilon^{1/p}} = \overline{\varepsilon}^{1/p} = 1$, and consequently obtain a Teichmüller expansion

$$\xi_\varepsilon = [m_0] + [m_1 + 1]p + \sum_{n \geq 2} [m_n]p^n \qquad \text{with } m_n \in \mathfrak{m}_F.$$

Since we have $|m_0| < 1$ and $|m_1 + 1| = 1$, we deduce by Proposition 1.1.12 that ξ_ε is a primitive element in A_{inf}. Moreover, ξ_ε is nondegenerate as we have

$$m_0 = 1 + \varepsilon^{1/p} + \cdots + \varepsilon^{(p-1)/p} = \frac{\varepsilon - 1}{\varepsilon^{1/p} - 1} \neq 0.$$

It is also evident that ξ_ε divides $[\varepsilon] - 1$. On the other hand, ξ_ε does not divide $[\varepsilon^{1/p}] - 1$, since otherwise $\xi_\varepsilon = 1 + [\varepsilon^{1/p}] + \cdots + [\varepsilon^{(p-1)/p}]$ should divide p, yielding a contradiction by Proposition 1.1.13. □

PROPOSITION 2.3.5. For every $\varepsilon \in 1 + \mathfrak{m}_F^*$, there exists some $y_\varepsilon \in Y$ with $\text{ord}_{y_\varepsilon}(\log(\varepsilon)) = 1$.

PROOF. Proposition 2.3.4 allows us to write $[\varepsilon] - 1 = \xi_\varepsilon([\varepsilon^{1/p}] - 1)$ for some nondegenerate primitive element $\xi_\varepsilon \in A_{\text{inf}}$ which does not divide $[\varepsilon^{1/p}] - 1$. Then by Example 2.2.9 and Lemma 2.2.10 we find an element $y_\varepsilon \in Y$ with $\text{ord}_{y_\varepsilon}([\varepsilon] - 1) = 1$. This means that the image of $[\varepsilon] - 1$ in $B_{\text{dR}}^+(y_\varepsilon)$ is a uniformizer. The assertion then follows from the fact that $\log(\varepsilon)$ is divisible by $[\varepsilon] - 1$ but not by $([\varepsilon] - 1)^2$. □

PROPOSITION 2.3.6. There exists a bijection $Y \stackrel{\sim}{\longrightarrow} (1 + \mathfrak{m}_F^*)/\mathbb{Z}_p^\times$ which maps the equivalence class of an untilt C of F to the $\mathbb{Z}_p^\times$-orbit of elements $\varepsilon_C \in 1 + \mathfrak{m}_F^*$ with $\varepsilon_C^\sharp = 1$ and $\left(\varepsilon_C^{1/p}\right)^\sharp \neq 1$.

PROOF. Let y be an arbitrary element in Y, represented by an untilt C of F. Choosing an element $\varepsilon_C \in 1 + \mathfrak{m}_F^*$ with $\varepsilon_C^\sharp = 1$ and $\left(\varepsilon_C^{1/p}\right)^\sharp \neq 1$ amounts to choosing a system of primitive p-power roots of unity in $C^\flat \simeq F$. Such a system exists uniquely up to $\mathbb{Z}_p^\times$-multiple by Proposition 1.1.6.

Let us now consider an arbitrary element $\varepsilon \in 1 + \mathfrak{m}_F^*$. Proposition 2.3.4 yields a nondegenerate primitive element $\xi_\varepsilon \in A_{\text{inf}}$ which divides $[\varepsilon] - 1$ but not $[\varepsilon^{1/p}] - 1$. Then by Theorem 1.1.21 we get an untilt C_ε of F with $\varepsilon^\sharp = 1$ and $\left(\varepsilon^{1/p}\right)^\sharp \neq 1$. Moreover, for every untilt C of F with $\varepsilon^\sharp = 1$ and $\left(\varepsilon^{1/p}\right)^\sharp \neq 1$, we have

$$0 = \frac{\varepsilon^\sharp - 1}{\left(\varepsilon^{1/p}\right)^\sharp - 1} = \frac{\theta_C([\varepsilon] - 1)}{\theta_C([\varepsilon^{1/p}] - 1)} = \theta_C(\xi_\varepsilon)$$

and consequently find by Proposition 1.1.19 and Theorem 1.1.21 that C and C_ε are equivalent. Therefore we deduce that ε is the image of a unique element in Y. □

Definition 2.3.7. Let φ_F denote the Frobenius automorphism of F.

(1) Given an untilt C of F with a continuous isomorphism $\iota : C^\flat \simeq F$, we define its *Frobenius twist* $\phi(C)$ as the perfectoid field C with the isomorphism $\varphi_F^n \circ \iota$.
(2) We define the *Frobenius action* on Y as the map $\phi : Y \to Y$ induced by Frobenius twists.

LEMMA 2.3.8. For every characteristic 0 untilt C of F we have $\widehat{\theta_{\phi(C)}} = \widehat{\theta_C} \circ \varphi$

PROOF. The identity is evident on $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ by construction. The assertion then follows by continuity. □

Remark. In Example 1.3.22 we described the Frobenius action ϕ on $\mathcal{Y}$. By Lemma 2.3.8 it is straightforward to check that the map $Y \longrightarrow \mathcal{Y}$ given by Example 1.3.8 is compatible with the Frobenius actions on Y and $\mathcal{Y}$.

PROPOSITION 2.3.9. Let f be a nonzero element in $B^{\varphi=p^n}$ for some $n \geq 0$. Then we have $\mathrm{ord}_y(f) = \mathrm{ord}_{\phi(y)}(f)$ for all $y \in Y$.

PROOF. Let C be an untilt of F which represents y. By corollary 2.2.4 there exists a primitive element ξ which generates $\ker(\widehat{\theta_C})$. It is then straightforward to check by Proposition 1.1.12 that $\varphi(\xi)$ is a primitive element in A_{inf}. Moreover, we have $\varphi(\xi) \in \ker(\widehat{\theta_{\phi(C)}})$ by Lemma 2.3.8. Let us write $i := \mathrm{ord}_y(f)$ and $j := \mathrm{ord}_{\phi(y)}(f)$. By Proposition 2.2.7 we may write

$$f = \xi^i g = \varphi(\xi)^j h \qquad \text{with } g, h \in B.$$

Then we have $f = p^{-n}\varphi(f) = \varphi(\xi)^i \cdot p^{-n} g$ and consequently find $i \leq j$. Similarly, we have $f = \varphi^{-1}(\varphi(f)) = p^n \varphi^{-1}(f) = \xi^j \cdot p^n h$ and consequently find $i \geq j$. Therefore we deduce $i = j$ as desired. □

PROPOSITION 2.3.10. For every $\varepsilon \in 1 + \mathfrak{m}_F^*$, there exists some $y_\varepsilon \in Y$ with

$$\mathrm{Div}(\log(\varepsilon)) = \sum_{n \in \mathbb{Z}} \phi^n(y_\varepsilon).$$

PROOF. Proposition 2.3.6 yields an untilt C_ε of F with $\varepsilon^{\sharp_{C_\varepsilon}} = 1$ and $\left(\varepsilon^{1/p}\right)^{\sharp_{C_\varepsilon}} \neq 1$. Let $y_\varepsilon \in Y$ be the equivalence class of C_ε. Consider an arbitrary element $y \in Y$, represented by an untilt C of F. We know by Proposition 3.3.11 in Chapter II that $\ker(\log_{\mu_{p^\infty}})$ is the torsion subgroup of $1 + \mathfrak{m}_C$ where $\mathfrak{m}_C$ denotes the maximal ideal of $\mathcal{O}_C$. Since we have $\varepsilon \neq 1$ by assumption, Proposition 2.3.3 implies that $\log(\varepsilon)$ vanishes at y if and only if there exists some $n \in \mathbb{Z}$ with $\left(\varepsilon^{p^n}\right)^{\sharp_C} = 1$ and $\left(\varepsilon^{p^{n-1}}\right)^{\sharp_C} \neq 1$, or equivalently $(\varphi_F^n(\varepsilon))^{\sharp_C} = 1$ and $\left(\varphi_F^{n-1}(\varepsilon)\right)^{\sharp_C} \neq 1$ where φ_F denotes the Frobenius automorphism of F. Hence by Proposition 2.3.6 we deduce that $\log(\varepsilon)$ vanishes at y if and only if there exists some $n \in \mathbb{Z}$ with $y = \phi^n(y_\varepsilon)$. Since we have $\log(\varepsilon) \in B^{\varphi=p}$, the assertion follows by Proposition 2.3.5 and Proposition 2.3.9. □

PROPOSITION 2.3.11. There exists a natural bijection $(1 + \mathfrak{m}_F^*)/\mathbb{Q}_p^\times \xrightarrow{\sim} Y/\phi^{\mathbb{Z}}$ which maps the $\mathbb{Q}_p^\times$-orbit of an element $\varepsilon \in 1 + \mathfrak{m}_F^*$ to the set of elements in Y at which $\log(\varepsilon)$ vanishes.

PROOF. Lemma 2.3.8 implies that the Frobenius action ϕ on Y corresponds to the multiplication by $1/p$ on $(1+\mathfrak{m}_F^*)/\mathbb{Z}_p^\times$ under the bijection $Y \xrightarrow{\sim} (1+\mathfrak{m}_F^*)/\mathbb{Z}_p^\times$ given by Proposition 2.3.6. Hence we obtain a natural bijection $(1 + \mathfrak{m}_F^*)/\mathbb{Q}_p^\times \xrightarrow{\sim} Y/\phi^{\mathbb{Z}}$. Let us now consider an arbitrary element $\varepsilon \in 1 + \mathfrak{m}_F^*$. Its $\mathbb{Q}_p^\times$-orbit maps to the ϕ-orbit of an element $y \in Y$ with a representative C that satisfies $\varepsilon^\sharp = 1$. Then we find $\widehat{\theta_C}(\log(\varepsilon)) = \log_{\mu_{p^\infty}}(\varepsilon^\sharp) = 0$ by Proposition 2.3.3, and consequently deduce the desired assertion by Proposition 2.3.10. □

2.4. Points and regularity

In this subsection, we prove that the Fargues-Fontaine curve is a Dedekind scheme whose closed points classify the Frobenius orbits in Y. For the rest of this chapter, let us write $P := \bigoplus B^{\varphi=p^n}$ and denote by $|X|$ the set of closed points in X. We also invoke the following technical result without proof.

PROPOSITION 2.4.1. Let f and g be elements in B. Then f is divisible by g in B if and only if we have $\mathrm{ord}_y(f) \geq \mathrm{ord}_y(g)$ for all $y \in Y$.

Remark. This is one of the most difficult results from the original work of Fargues and Fontaine [**FF18**]. Curious readers can find a complete proof in [**Lur**, Lecture 13-16]. Here we provide a brief sketch of the proof.

We only need to prove the if part as the converse is obvious by Lemma 2.2.10. Moreover, in light of Lemma 2.2.2 we may replace B by $B_{[a,b]}$ for an arbitrary interval $[a,b] \subseteq (0,1)$. The key point is to show that every element in $B_{[a,b]}$ admits a (necessarily unique) factorization into primitive elements. By a similar argument as in Proposition 2.2.11 the proof boils down to showing that every $h \in B_{[a,b]}$ with $\partial_-\mathcal{L}_{h,[a,b]}(s) \neq \partial_+\mathcal{L}_{h,[a,b]}(s)$ for some $s \in [-\log_p(b), -\log_p(a)]$ has a zero $y \in Y_{[p^{-s},p^{-s}]}$.

Let us set $\widehat{Y} := Y \cup \{ o \}$, where o denotes the equivalence class of F as the trivial untilt of itself. Then $\widehat{Y}$ turns out to be complete with respect to an ultrametric d given by

$$d(y_1, y_2) := |\theta_{C_2}(\xi_1)|_{C_2} \qquad \text{for all } y_1, y_2 \in \widehat{Y}$$

where ξ_1 and C_2 respectively denote a primitive element that vanishes at y_1 and an untilt of F that represents y_2. If h is an element in $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$, an elegant approximation argument using Legendre-Newton polygons allows us to construct a zero $y \in Y_{[p^{-s},p^{-s}]}$ of h as the limit of a Cauchy sequence (y_n) in $\widehat{Y}$ with $|y_n| = p^{-s}$ and $\lim\limits_{n\to\infty} |h(y_n)|_{C_n} = 0$ where each C_n is a representative of y_n. For the general case, we can construct Cauchy sequences (h_n) in $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ and (y_n) in $Y_{[p^{-s},p^{-s}]}$ with $h_n(y_n) = 0$ and $\lim\limits_{n\to\infty} h_n = h$ with respect to the Gauss p^{-s}-norm, thereby obtaining a zero $y \in Y_{[p^{-s},p^{-s}]}$ of h as the limit of (y_n).

COROLLARY 2.4.2. The ring $B^{\varphi=1}$ is a field.

PROOF. Consider an arbitrary nonzero element $f \in B^{\varphi=1}$. We have $\mathrm{Div}(f) = 0$, since otherwise f would be divisible by some $g \in B^{\varphi=1/p}$, thereby contradicting Proposition 2.1.15. Hence by Proposition 2.4.1 we deduce that f admits an inverse in $B^{\varphi=1}$ as desired. □

Remark. As remarked after Proposition 2.1.15, we will see in Proposition 3.1.6 that $B^{\varphi=1}$ is canonically isomorphic to $\mathbb{Q}_p$.

LEMMA 2.4.3. Let f be an element in $B^{\varphi=p^n}$ for some $n \geq 0$, and let ε be an element in $1 + \mathfrak{m}_F^*$. Assume that both f and $\log(\varepsilon)$ vanish at some $y \in Y$. Then there exists some $g \in B^{\varphi=p^{n-1}}$ with $f = \log(\varepsilon)g$.

PROOF. By Proposition 2.3.9 we have

$$\mathrm{ord}_{\phi^i(y)}(f) = \mathrm{ord}_y(f) \geq 1 \qquad \text{for all } i \in \mathbb{Z}.$$

In addition, by Proposition 2.3.10 we find

$$\mathrm{Div}(\log(\varepsilon)) = \sum_{i \in \mathbb{Z}} \phi^i(y).$$

Since $\log(\varepsilon)$ belongs to $B^{\varphi=p}$ by construction, the assertion follows by Proposition 2.4.1. □

PROPOSITION 2.4.4. For every $\varepsilon \in 1 + \mathfrak{m}_F$, the element $\log(\varepsilon) \in B^{\varphi=p}$ is a prime in P.

PROOF. The assertion is obvious for $\varepsilon = 1$ as P is an integral domain by Corollary 2.1.17. We henceforth assume $\varepsilon \neq 1$. Consider arbitrary elements f and g in P such that $\log(\varepsilon)$ divides fg in P. We wish to show that $\log(\varepsilon)$ divides either f or g in P. Since $\log(\varepsilon)$ is homogeneous, we may assume without loss of generality that both f and g are homogeneous. Proposition 2.3.5 implies that $\log(\varepsilon)$ vanishes at some $y_\varepsilon \in Y$. Then we find by Lemma 2.2.10 that either f or g vanishes at y_ε, and in turn deduce the desired assertion by Lemma 2.4.3. □

PROPOSITION 2.4.5. Let f be a nonzero element in $B^{\varphi=p^n}$ for some $n \geq 0$.

(1) The map φ uniquely extends to an automorphism $\varphi_{1/f}$ on $B[1/f]$.
(2) We may write
$$f = \lambda \log(\varepsilon_1) \cdots \log(\varepsilon_n) \qquad \text{with } \lambda \in B^{\varphi=1} \text{ and } \varepsilon_i \in 1 + \mathfrak{m}_F^* \tag{2.10}$$
where the factors are uniquely determined up to $\mathbb{Q}_p^\times$-multiple.

PROOF. The first statement is straightforward to verify. Let us prove the second statement by induction on n. Since the assertion is obvious for $n = 0$, we henceforth assume $n > 0$. Then f vanishes at some $y \in Y$; otherwise, it would be invertible in B by Proposition 2.4.1 and thus would yield a nonzero element $f^{-1} \in B^{\varphi=p^{-n}}$, contradicting Proposition 2.1.15. Now Lemma 2.4.3 and Proposition 2.3.11 together yield some $\varepsilon_n \in 1 + \mathfrak{m}_F$ and $g \in B^{\varphi=p^{n-1}}$ with $f = \log(\varepsilon_n)g$. Hence by induction hypothesis we obtain an expression as in (2.10), where the factors are uniquely determined up to $\mathbb{Q}_p^\times$-multiple by Proposition 2.4.4. □

Definition 2.4.6. Given a nonzero homogeneous element $f \in P$, we refer to the map $\varphi_{1/f}$ described in Proposition 2.4.5 as the *Frobenius automorphism* of $B[1/f]$. We often abuse notation and write φ instead of $\varphi_{1/f}$.

PROPOSITION 2.4.7. Every non-generic point $x \in X$ is a closed point, induced by a prime $\log(\varepsilon)$ in P for some $\varepsilon \in 1 + \mathfrak{m}_F^*$. Moreover, its residue field is naturally isomorphic to the perfectoid field given by any $y \in Y$ at which $\log(\varepsilon)$ vanishes.

PROOF. By Proposition 2.4.5 there exists a nonzero element $t \in B^{\varphi=p}$ such that x lies in the open subscheme $\operatorname{Spec}(B[1/t]^{\varphi=1})$ of $X = \operatorname{Proj}(P)$. Let us denote by $\mathfrak{p}$ the prime ideal of $B[1/t]^{\varphi=1}$ which corresponds to x, and take an element $f/t^n \in \mathfrak{p}$ with $f \in B^{\varphi=p^n}$. By Proposition 2.4.5 we may write
$$\frac{f}{t^n} = \lambda \cdot \frac{\log(\varepsilon_1)}{t} \cdot \frac{\log(\varepsilon_2)}{t} \cdots \frac{\log(\varepsilon_n)}{t} \qquad \text{with } \lambda \in B^{\varphi=1} \text{ and } \varepsilon_i \in 1 + \mathfrak{m}_F^*.$$
Since λ is a unit in $B^{\varphi=1}$ by Corollary 2.4.2, we have $\log(\varepsilon)/t \in \mathfrak{p}$ for some $\varepsilon \in 1 + \mathfrak{m}_F^*$.

Take an element $y \in Y$ at which $\log(\varepsilon)$ vanishes, and choose a representative C of y. Then t does not vanish at y, since otherwise Corollary 2.4.2 and Lemma 2.4.3 together would imply that $\log(\varepsilon)/t$ is an invertible element in $B^{\varphi=1}$, which is impossible as $\mathfrak{p}$ is a prime ideal. We thus obtain a map $\theta_x : B[1/t]^{\varphi=1} \hookrightarrow B[1/t] \twoheadrightarrow C$ where the second arrow is induced by $\widehat{\theta_C}$.

It suffices to show that θ_x is a surjective map whose kernel is generated by $\log(\varepsilon)/t$. Proposition 2.3.3 implies that $\widehat{\theta_C}$ induces a surjection $B^{\varphi=p} \twoheadrightarrow C$, which in turn implies that θ_x is already surjective when restricted to $(1/t)B^{\varphi=p}$. Let us now consider an arbitrary element $f'/t^n \in \ker(\theta_x)$ with $f' \in B^{\varphi=p^n}$. Arguing as in the first paragraph, we find that f'/t^n is divisible by $\log(\varepsilon')/t \in \ker(\theta_x)$ for some $\varepsilon' \in 1 + \mathfrak{m}_F^*$. Then we have $\widehat{\theta_C}(\log(\varepsilon')) = 0$, which means that $\log(\varepsilon')$ vanishes at y. Therefore we deduce by Lemma 2.4.3 that $\log(\varepsilon)/t$ divides $\log(\varepsilon')/t$, and thus divides f'/t as desired. □

THEOREM 2.4.8 (Fargues-Fontaine [**FF18**]). The scheme X has the following properties:

(i) There exists a natural bijection $|X| \xrightarrow{\sim} Y/\phi^{\mathbb{Z}}$ which maps the point induced by $\log(\varepsilon)$ for some $\varepsilon \in 1+\mathfrak{m}_F^*$ to the set of elements in Y at which $\log(\varepsilon)$ vanishes.

(ii) X is a Dedekind scheme such that the open subscheme $X \setminus \{ x \}$ for every $x \in |X|$ is the spectrum of a principal ideal domain.

(iii) For every $x \in |X|$, its completed local ring $\widehat{\mathcal{O}_{X,x}}$ admits a natural identification

$$\widehat{\mathcal{O}_{X,x}} \cong B_{\text{dR}}^+(y)$$

where y is any element in the image of x under the bijection $|X| \xrightarrow{\sim} Y/\phi^{\mathbb{Z}}$.

PROOF. Proposition 2.4.7 yields a surjective map $1+\mathfrak{m}_F^* \twoheadrightarrow |X|$ which associates to each $\varepsilon \in 1+\mathfrak{m}_F^*$ the point $x \in X$ induced by the prime $\log(\varepsilon) \in P$. Moreover, Lemma 2.4.3 implies that two elements ε_1 and ε_2 in $1+\mathfrak{m}_F^*$ map to the same point in $|X|$ if and only if $\log(\varepsilon_1)$ and $\log(\varepsilon_2)$ have a common zero. Therefore we deduce the property (i) by Proposition 2.3.11.

Let us now fix a closed point x in X. As shown in the preceding paragraph, the point x is induced by $\log(\varepsilon)$ for some $\varepsilon \in 1+\mathfrak{m}_F^*$. It follows that $X \setminus \{ x \}$ is the spectrum of the ring $B[1/\log(\varepsilon)]^{\varphi=1}$. In addition, we find by Proposition 2.4.7 that every prime ideal of $B[1/\log(\varepsilon)]^{\varphi=1}$ is a principal ideal. Therefore we obtain the property (ii) by a general fact as stated in [**Sta**, Tag 05KH].

It remains to establish the property (iii). Let us fix an element $y \in Y$ at which $\log(\varepsilon)$ vanishes, and take an untilt C of F which represents y. We also choose an element $t \in B^{\varphi=p}$ which is not divisible by $\log(\varepsilon)$. Then we have a surjective map $\widehat{\theta_C}[1/t] : B[1/t] \twoheadrightarrow C$ induced by $\widehat{\theta_C}$. Let us denote by θ_x the restriction of $\widehat{\theta_C}[1/t]$ to $B[1/t]^{\varphi=1}$. Proposition 2.4.7 implies that we may identify x as a point in $\operatorname{Spec}(B[1/t]^{\varphi=1})$ given by $\ker(\theta_x)$. Hence we obtain an identification

$$\widehat{\mathcal{O}_{X,x}} \cong \varprojlim_j B[1/t]^{\varphi=1}/\ker(\theta_x)^j. \tag{2.11}$$

Meanwhile, Proposition 2.2.7 allows us to identify $B_{\text{dR}}^+(y)$ as the completed local ring of a closed point $\widehat{y} \in \operatorname{Spec}(B)$ given by $\ker(\widehat{\theta_C})$, thereby yielding an identification

$$B_{\text{dR}}^+(y) \cong \varprojlim_j B[1/t]/\ker(\widehat{\theta_C}[1/t])^j. \tag{2.12}$$

For an arbitrary element $f/t^n \in B[1/t]^{\varphi=1} \cap \ker(\widehat{\theta_C})^j$ with $f \in B^{\varphi=p^n}$ and $j \geq 1$, we have $\operatorname{ord}_y(f) \geq j$ and consequently find by Lemma 2.4.3 that f/t^n is divisible by $\log(\varepsilon)^j/t^j$. Since $\log(\varepsilon)/t$ belongs to $\ker(\theta_x)$, we obtain an identification

$$B[1/t]^{\varphi=1} \cap \ker(\widehat{\theta_C})^j = \ker(\theta_x)^j \qquad \text{for all } j \geq 1$$

and in turn get a natural injective map

$$\varprojlim_j B[1/t]^{\varphi=1}/\ker(\theta_x)^j \hookrightarrow \varprojlim_j B[1/t]/\ker(\widehat{\theta_C}[1/t])^j. \tag{2.13}$$

Moreover, since both $B[1/t]^{\varphi=1}/\ker(\theta_x)$ and $B[1/t]/\ker(\widehat{\theta_C}[1/t])$ are isomorphic to C, the map (2.13) is surjective by a general fact as stated in [**Sta**, Tag 0315]. Therefore we obtain the property (iii) by (2.11) and (2.12). □

Remark. The scheme X is defined over $\mathbb{Q}_p$ as we will see in Corollary 3.1.7. However, it is not of finite type over $\mathbb{Q}_p$ since the residue field of an arbitrary closed point is an infinite extension of $\mathbb{Q}_p$ by Proposition 2.4.7.

3. Vector bundles

Our main objective in this section is to discuss several key properties of vector bundles on the Fargues-Fontaine curve. The primary references for this section are Fargues and Fontaine's survey paper [**FF14**] and Lurie's notes [**Lur**].

3.1. Frobenius eigenspaces

In order to study the vector bundles on X, it is crucial to understand the structure of the graded ring $P = \bigoplus B^{\varphi = p^n}$. In this subsection, we aim to establish an explicit description of the Frobenius eigenspaces $B^{\varphi = p^n}$ for all $n \geq 0$.

PROPOSITION 3.1.1. The natural map $F \longrightarrow B$ given by Teichmüller lifts is continuous.

PROOF. Take a characteristic 0 untilt C of F. The natural map $F \longrightarrow B$ composed with $\widehat{\theta_C}$ coincides with the sharp map associated to C, which is evidently continuous by construction. Hence the assertion follows by Proposition 1.2.16. □

LEMMA 3.1.2. For every $f \in B$ with $|f|_\rho \leq 1$ for all $\rho \in (0,1)$, there exists a sequence (f_n) in $A_{\inf}[1/[\varpi]]$ which converges to f with respect to all Gauss norms.

PROOF. We may assume $f \neq 0$, since the assertion is obvious for $f = 0$. Take a sequence $(\widetilde{f_n})$ in $A_{\inf}[1/p, 1/[\varpi]]$ which converges to f with respect to all Gauss norms. For each $n \geq 1$, we may write $\widetilde{f_n} = f_n + \sum_{i<0} [c_{n,i}] p^i$ with $c_{n,i} \in F$ and $f_n \in A_{\inf}[1/[\varpi]]$. Take arbitrary real numbers $\rho \in (0,1)$ and $\epsilon > 0$. Then for all sufficiently large n we have

$$\left| \widetilde{f_n} - f_n \right|_\rho = \sup_{i<0} \left(|c_{n,i}| \rho^i \right) \leq \sup_{i<0} \left(\epsilon^{-i} \right) \cdot \sup_{i<0} \left(|c_{n,i}| \epsilon^i \rho^i \right) \leq \epsilon \cdot \left| \widetilde{f_n} \right|_{\epsilon\rho} = \epsilon \, |f|_{\epsilon\rho} \leq \epsilon$$

where the second identity follows from Lemma 2.1.8. Hence we obtain $\lim_{n\to\infty} \left| \widetilde{f_n} - f_n \right|_\rho = 0$ for all $\rho \in (0,1)$, thereby deducing that (f_n) converges to f with respect to all Gauss norms. □

PROPOSITION 3.1.3. Let f be an element in B. Assume that there exists an integer $n \geq 0$ with $|f|_\rho \leq \rho^n$ for all $\rho \in (0,1)$. Then we may write $f = [c]p^n + g$ for some $c \in \mathcal{O}_F$ and $g \in B$ with $|g|_\rho \leq \rho^{n+1}$ for all $\rho \in (0,1)$.

PROOF. We may replace f by f/p^n to assume $n = 0$. Lemma 3.1.2 yields a sequence (f_i) in $A_{\inf}[1/[\varpi]]$ which converges to f with respect to all Gauss norms. For each $i \geq 1$, we denote by $[c_i]$ the first coefficient in the Teichmüller expansion of f_i. Then we have $|c_{i+1} - c_i| \leq |f_{i+1} - f_i|_\rho$ for all $i \geq 1$ and $\rho \in (0,1)$. This means that the sequence (c_i) is Cauchy in F and thus converges to an element $c \in F$. In addition, given a real number $\rho \in (0,1)$, Lemma 2.1.8 yields $|c_i| \leq |f_i|_\rho = |f|_\rho \leq 1$ for all sufficiently large i, thereby implying $c \in \mathcal{O}_F$.

Let us now set $g_i := f_i - [c_i] \in A_{\inf}[1/[\varpi]]$ for each $i \geq 1$ and take $g := f - [c] \in B$. We may assume $g \neq 0$, since the assertion is obvious if we have $g = 0$. Each g_i admits a Teichmüller expansion where only positive powers of p occur, so that all slopes of $\mathcal{L}_{g_i}$ are positive integers by Proposition 2.1.7. Moreover, Proposition 3.1.1 implies that the sequence (g_i) converges to g with respect to all Gauss norms. Therefore we deduce by Lemma 2.1.8 that all slopes of $\mathcal{L}_g$ are positive integers. We then use Lemma 2.1.2 to obtain

$$\mathcal{L}_g(s) \geq \min \left(\mathcal{L}_f(s), \mathcal{L}_{[c]}(s) \right) = \min \left(-\log_p \left(|f|_{p^{-s}} \right), -\log_p \left(|c| \right) \right) \geq 0 \qquad \text{for all } s > 0,$$

thereby deducing $\mathcal{L}_g(s) \geq s$ for all $s > 0$, or equivalently $|g|_\rho \leq \rho$ for all $\rho \in (0,1)$. □

Proposition 3.1.4. Let f be a nonzero element in B.

(1) The element f belongs to A_{inf} if and only if we have $|f|_\rho \leq 1$ for all $\rho \in (0,1)$.
(2) The element f belongs to $A_{\text{inf}}[1/p]$ if and only if there exists an integer n with $|f|_\rho \leq \rho^n$ for all $\rho \in (0,1)$.
(3) The element f belongs to $A_{\text{inf}}[1/[\varpi]]$ if and only if there exists a constant $C > 0$ with $|f|_\rho \leq C$ for all $\rho \in (0,1)$.
(4) The element f belongs to $A_{\text{inf}}[1/p, 1/[\varpi]]$ if and only if there exist a constant $C > 0$ and an integer n with $|f|_\rho \leq C\rho^n$ for all $\rho \in (0,1)$.

Proof. If f belongs to A_{inf}, then we clearly have $|f|_\rho \leq 1$ for all $\rho \in (0,1)$. Conversely, if we have $|f|_\rho \leq 1$ for all $\rho \in (0,1)$, then by Proposition 3.1.3 we can inductively construct a sequence (c_i) in $\mathcal{O}_F$ with

$$\left| f - \sum_{i=0}^{n-1} [c_i] p^i \right|_\rho \leq \rho^n \qquad \text{for all } n \geq 0 \text{ and } \rho \in (0,1),$$

thereby deducing $f \in A_{\text{inf}}$. Therefore we establish the statement (1).

Now we find that f belongs to $A_{\text{inf}}[1/p]$ if and only if there exists an integer n with $p^n f \in A_{\text{inf}}$, or equivalently $|f|_\rho \leq |p|_\rho^{-n} = \rho^{-n}$ for all $\rho \in (0,1)$, thereby obtaining the statement (2). Similarly, we find that f belongs to $A_{\text{inf}}[1/[\varpi]]$ if and only if there exists an integer n with $[\varpi^n] f \in A_{\text{inf}}$, or equivalently $|f|_\rho \leq |[\varpi]|_\rho^{-n} = |\varpi|^{-n}$ for all $\rho \in (0,1)$, thereby obtaining the statement (3). Finally, we find that f belongs to $A_{\text{inf}}[1/p, 1/[\varpi]]$ if and only if there exist integers l and n with $p^n [\varpi]^l f \in A_{\text{inf}}$, or equivalently $|f|_\rho \leq \left|[\varpi]^l p^n\right|_\rho = |\varpi|^l \rho^n$ for all $\rho \in (0,1)$, thereby obtaining the statement (4). $\square$

Lemma 3.1.5. Given a nonzero element $f \in B^{\varphi=1}$, there exists an integer n with $|f|_\rho = \rho^n$ for all $\rho \in (0,1)$.

Proof. By Lemma 2.1.14 we have

$$p\mathcal{L}_f(s) = \mathcal{L}_{\varphi(f)}(ps) = \mathcal{L}_f(ps) \qquad \text{for all } s > 0, \tag{3.1}$$

and consequently find $p\partial_+ \mathcal{L}_f(s) = p\partial_+ \mathcal{L}_f(ps)$ for all $s > 0$. Hence Corollary 2.1.11 implies that $\mathcal{L}_f$ is linear with integer slope, which means that there exist an integer n and a real number r with $\mathcal{L}_f(s) = ns + r$ for all $s > 0$. We then find $r = 0$ by (3.1), and in turn obtain $\mathcal{L}_f(s) = ns$ for all $s > 0$, or equivalently $|f|_\rho = \rho^n$ for all $\rho \in (0,1)$. $\square$

Proposition 3.1.6. The ring $B^{\varphi=1}$ is canonically isomorphic to $\mathbb{Q}_p$.

Proof. Let $W(\mathbb{F}_p)$ denote the ring of Witt vectors over $\mathbb{F}_p$. Under the identification

$$\mathbb{Q}_p \cong W(\mathbb{F}_p)[1/p] \cong \left\{ \sum [c_n] p^n \in A_{\text{inf}}[1/p] : c_n \in \mathbb{F}_p \right\}, \tag{3.2}$$

we may regard $\mathbb{Q}_p$ as a subring of $B^{\varphi=1}$. Let us now consider an arbitrary nonzero element $f \in B^{\varphi=1}$. Proposition 3.1.4 and Lemma 3.1.5 together imply that f is an element in $A_{\text{inf}}[1/p]$. Hence we may write $f = \sum [c_n] p^n$ with $c_n \in \mathcal{O}_F$. Since f is invariant under φ, for each $n \in \mathbb{Z}$ we find $c_n^p = c_n$, or equivalently $c_n \in \mathbb{F}_p$. We thus deduce $f \in \mathbb{Q}_p$ under the identification (3.2), thereby completing the proof. $\square$

Remark. Our proof does not depend on Proposition 2.4.1 that we assume without proof.

Corollary 3.1.7. The scheme X is defined over $\mathbb{Q}_p$.

PROPOSITION 3.1.8. The map $\log : 1 + \mathfrak{m}_F \longrightarrow B^{\varphi=p}$ is a continuous $\mathbb{Q}_p$-linear isomorphism.

PROOF. Choose a characteristic 0 untilt C of F. The sharp map associated to C is continuous by construction. In addition, the map $\log_{\mu_{p^\infty}}$ is continuous by Proposition 3.3.11 in Chapter II. Therefore it follows by Proposition 2.3.3 and Proposition 1.2.16 that the map log is continuous. Moreover, since every element in $\mathbb{Q}_p$ is the limit of a sequence in $\mathbb{Z}$, we obtain the $\mathbb{Q}_p$-linearity of log by Proposition 2.3.1, and consequently deduce the surjectivity of log by Proposition 2.4.5 and Proposition 3.1.6. We also find that log is injective, as Proposition 2.3.10 yields $\log(\varepsilon) \neq 0$ for every $\varepsilon \in 1 + \mathfrak{m}_F^*$. Therefore we establish the desired assertion. ☐

COROLLARY 3.1.9. There exists a natural bijection $|X| \xrightarrow{\sim} (B^{\varphi=p} \setminus \{ 0 \})/\mathbb{Q}_p^\times$ which maps the point induced by $\log(\varepsilon)$ for some $\varepsilon \in 1 + \mathfrak{m}_F^*$ to the $\mathbb{Q}_p^\times$-orbit of $\log(\varepsilon)$ in $B^{\varphi=p}$.

PROOF. This is merely a restatement of the property (i) in Theorem 2.4.8 using Proposition 3.1.8. ☐

COROLLARY 3.1.10. Let f be a nonzero element in $B^{\varphi=p^n}$ for some $n \geq 1$. We may write

$$f = \log(\varepsilon_1)\log(\varepsilon_2)\cdots\log(\varepsilon_n) \qquad \text{with } \varepsilon_i \in 1 + \mathfrak{m}_F^*$$

where the factors are uniquely determined up to $\mathbb{Q}_p^\times$-multiple.

PROOF. This is an immediate consequence of Proposition 3.1.6, Proposition 3.1.8, and Proposition 2.4.5. ☐

Remark. Corollary 3.1.9 and Corollary 3.1.10 are respectively analogues of the following facts about the complex projective line $\mathbb{P}^1_{\mathbb{C}} = \mathrm{Proj}\,(\mathbb{C}[z_1, z_2])$:

(1) Closed points in $\mathbb{P}^1_{\mathbb{C}}$ are in bijection with the $\mathbb{Q}_p$-orbits of linear homogeneous polynomials in $\mathbb{C}[z_1, z_2]$.
(2) Every homogeneous polynomial in $\mathbb{C}[z_1, z_2]$ of positive degree admits a unique factorization into linear homogeneous polynomials up to $\mathbb{C}^\times$-multiple

It is therefore reasonable to expect that the Fargues-Fontaine curve X is geometrically similar to $\mathbb{P}^1_{\mathbb{C}}$, even though X is not of finite type over $\mathbb{Q}_p$. We will solidify this idea in the next subsection by studying line bundles on the Fargues-Fontaine curve.

PROPOSITION 3.1.11. Let B^+ be the closure of $A_{\mathrm{inf}}[1/p]$ in B. For every $n \in \mathbb{Z}$ we have $B^{\varphi=p^n} \subseteq B^+$.

PROOF. For $n \leq 0$, the assertion is obvious by Proposition 2.1.15 and Proposition 3.1.6. Moreover, we find

$$\log(\varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon] - 1)^n}{n} \in B^+ \qquad \text{for every } \varepsilon \in 1 + \mathfrak{m}_F$$

as each summand belongs to $A_{\mathrm{inf}}[1/p]$, thereby deducing the assertion for $n \geq 1$ by Corollary 3.1.10. ☐

Remark. For every nonzero element $f \in B^{\varphi=n}$, we find $\lim_{s \to 0} \mathcal{L}_f(s) = 0$ by the functional equation $p\mathcal{L}_f(s) = ns + \mathcal{L}_f(ps)$ as obtained in the proof of Proposition 2.1.15. Hence we can alternatively deduce Proposition 3.1.11 from an identification

$$B^+ = \left\{ f \in B : \lim_{s \to 0} \mathcal{L}_f(s) \geq 0 \right\}$$

which is not hard to verify using Proposition 2.1.9 and Proposition 3.1.4. We note that this proof does not rely on Proposition 2.4.1 which we assume without proof.

3.2. Line bundles and their cohomology

In this subsection, we classify and study line bundles on the Fargues-Fontaine curve. Throughout this subsection, we denote by $\mathrm{Div}(X)$ the group of Weil divisors on X, and by $\mathrm{Pic}(X)$ the Picard group of X. In addition, for every rational section f on X we write $\mathrm{Div}(f)$ for its associated Weil divisor on X.

Definition 3.2.1. We define the *divisor degree map* of X to be the group homomorphism $\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z}$ with $\deg(x) = 1$ for all $x \in |X|$.

PROPOSITION 3.2.2. For every $D \in \mathrm{Div}(X)$, we have $\deg(D) = 0$ if and only if D is principal.

PROOF. Let $K(X)$ denote the function field of X. We also let Q denote the fraction field of P. Note that there exists a natural identification

$$K(X) \cong \left\{ f/g \in Q : f, g \in B^{\varphi=p^n} \text{ for some } n \geq 0 \right\}. \tag{3.3}$$

Consider an arbitrary element $f \in K(X)^\times$. By (3.3) and Corollary 3.1.10 there exist some nonzero elements $t_1, t_2, \cdots, t_{2n} \in B^{\varphi=p}$ with

$$f = \frac{t_1 t_2 \cdots t_n}{t_{n+1} t_{n+2} \cdots t_{2n}}.$$

We then find $\deg(\mathrm{Div}(f)) = 0$ as Corollary 3.1.9 yields $x_1, x_2, \cdots, x_{2n} \in |X|$ with $\mathrm{Div}(t_i) = x_i$.

Let us now consider an arbitrary Weil divisor D on X with $\deg(D) = 0$. We may write

$$D = (x_1 + x_2 + \cdots + x_n) - (x_{n+1} + x_{n+2} + \cdots + x_{2n}) \qquad \text{with } x_i \in |X|.$$

Moreover, Corollary 3.1.9 yields $t_1, t_2, \cdots, t_{2n} \in B^{\varphi=p}$ with $\mathrm{Div}(t_i) = x_i$. Hence we have

$$D = \mathrm{Div}\left(\frac{t_1 t_2 \cdots t_n}{t_{n+1} t_{n+2} \cdots t_{2n}} \right),$$

which is easily seen to be a principal divisor by (3.3). □

Definition 3.2.3. For every $d \in \mathbb{Z}$, we write $P(d) := \bigoplus_{n \in \mathbb{Z}} B^{\varphi=p^{d+n}}$ and define the *d-th twist of* $\mathcal{O}_X$ to be the quasicoherent sheaf $\mathcal{O}(d)$ on X associated to $P(d)$.

LEMMA 3.2.4. For every $d \in \mathbb{Z}$, the sheaf $\mathcal{O}(d)$ is a line bundle on X with a canonical isomorphism $\mathcal{O}(d) \cong \mathcal{O}(1)^{\otimes d}$.

PROOF. The assertion follows from Corollary 3.1.10 by a general fact as stated in [**Sta**, Tag 01MT]. □

PROPOSITION 3.2.5. The divisor degree map of X induces a natural isomorphism $\mathrm{Pic}(X) \cong \mathbb{Z}$ whose inverse maps each $d \in \mathbb{Z}$ to the isomorphism class of $\mathcal{O}(d)$.

PROOF. Since X is a Dedekind scheme as noted in Theorem 2.4.8, we can identify $\mathrm{Pic}(X)$ with the class group of X. Hence by Proposition 3.2.2 the divisor degree map of X induces a natural isomorphism $\mathrm{Pic}(X) \cong \mathbb{Z}$. Let us now choose a nonzero element $t \in B^{\varphi=p}$, which induces a closed point x on X by Corollary 3.1.9. It is straightforward to check that t is a global section of $\mathcal{O}(1)$, which in turn implies by Lemma 3.2.4 that $\mathcal{O}(1)$ is isomorphic to the line bundle that arises from the Weil divisor $\mathrm{Div}(t) = x$ on X. Hence the isomorphism class of $\mathcal{O}(1)$ maps to $\deg(x) = 1$ under the isomorphism $\mathrm{Pic}(X) \cong \mathbb{Z}$. The assertion now follows by Lemma 3.2.4. □

Remark. Proposition 3.2.5 is an analogue of the fact that there exists a natural isomorphism $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}}) \cong \mathbb{Z}$ whose inverse maps each $d \in \mathbb{Z}$ to the isomorphism class of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d)$.

PROPOSITION 3.2.6. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a graded P-module, and let $\widetilde{M}$ be the associated quasicoherent $\mathcal{O}_X$-module. There exists a canonical functorial $\mathbb{Q}_p$-linear map $M_0 \longrightarrow H^0(X, \widetilde{M})$.

PROOF. Since we have $B^{\varphi=1} \cong \mathbb{Q}_p$ as noted in Proposition 3.1.6, the assertion follows by a general fact as stated in [**Sta**, Tag 01M7]. $\square$

Definition 3.2.7. Given a graded P-module M, we refer to the map $M_0 \longrightarrow H^0(X, \widetilde{M})$ in Proposition 3.2.6 as the *saturation map* for M.

PROPOSITION 3.2.8. Let d be a nonnegative integer, and let t be a nonzero element in $B^{\varphi=p}$. The multiplication by t on P induces a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^{\varphi=p^d} & \longrightarrow & B^{\varphi=p^{d+1}} & \longrightarrow & B^{\varphi=p^{d+1}}/tB^{\varphi=p^d} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \wr\downarrow & & \\ 0 & \longrightarrow & H^0(X, \mathcal{O}(d)) & \longrightarrow & H^0(X, \mathcal{O}(d+1)) & \longrightarrow & H^0(X, \mathcal{O}(d+1)/t\mathcal{O}(d)) & \longrightarrow & 0 \end{array}$$

where the vertical arrows respectively represent the saturation maps for $P(d)$, $P(d+1)$ and $P(d+1)/tP(d)$. Moreover, $\mathcal{O}(d+1)/t\mathcal{O}(d)$ is supported at the point $x \in |X|$ induced by t.

PROOF. Since P is an integral domain by Corollary 2.1.17, the multiplication by t on P yields an exact sequence of graded P-modules

$$0 \longrightarrow P(d) \xrightarrow{f \mapsto ft} P(d+1) \longrightarrow P(d+1)/tP(d) \longrightarrow 0 \tag{3.4}$$

which gives rise to an exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}(d) \longrightarrow \mathcal{O}(d+1) \longrightarrow \mathcal{O}(d+1)/t\mathcal{O}(d) \longrightarrow 0. \tag{3.5}$$

The top row of the diagram is induced by the sequence (3.4), and is exact. The bottom row of the diagram is induced by the sequence (3.5), and is left exact. The commutativity of the diagram is evident by the functoriality of saturation maps as noted in Proposition 3.2.6.

By Corollary 3.1.8 we may write $t = \log(\varepsilon)$ for some $\varepsilon \in 1 + \mathfrak{m}_F^*$. In addition, Proposition 2.3.10 yields an element $y \in Y$ at which t vanishes. Let us choose a representative C of y. Proposition 2.3.3 implies that $\widehat{\theta_C}$ restricts to a surjective map $B^{\varphi=p} \twoheadrightarrow C$. Hence for every $a \in C$ we can take $s_0, s \in B^{\varphi=p}$ with $\widehat{\theta_C}(s_0) = 1$ and $\widehat{\theta_C}(s) = a$, and consequently obtain $\widehat{\theta_C}(s_0^d s) = a$. In particular, the map $\widehat{\theta_C}$ restricts to a surjective map $B^{\varphi=p^{d+1}} \twoheadrightarrow C$. We also find by Lemma 2.4.3 that the kernel of this map is given by $tB^{\varphi=p^d}$. Therefore the map $\widehat{\theta_C}$ induces an isomorphism

$$B^{\varphi=p^{d+1}}/tB^{\varphi=p^d} \simeq C. \tag{3.6}$$

Let us now take $x \in |X|$ induced by t. Then Proposition 2.4.7 allows us to identify C with the residue field of x. In addition, Proposition 3.2.5 implies that $\mathcal{O}(d)$ and $\mathcal{O}(d+1)$ are respectively isomorphic to the line bundles that arise from the Weil divisors dx and $(d+1)x$. It is then straightforward to verify that $\mathcal{O}(d+1)/t\mathcal{O}(d)$ is supported at x with the stalk given by $t^{-d-1}\mathcal{O}_{X,x}/t^{-d}\mathcal{O}_{X,x} \simeq C$. This means that $\mathcal{O}(d+1)/t\mathcal{O}(d)$ is isomorphic to the skyscraper sheaf at x with value C. Furthermore, by (3.6) we obtain an isomorphism

$$B^{\varphi=p^{d+1}}/tB^{\varphi=p^d} \simeq C \cong H^0(X, \mathcal{O}(d+1)/t\mathcal{O}(d)),$$

which is easily seen to coincide with the saturation map for $P(d+1)/tP(d)$. We then deduce by the commutativity of the second square that the bottom row is exact, thereby completing the proof. $\square$

THEOREM 3.2.9 (Fargues-Fontaine [**FF18**]). We have the following facts about the cohomology of line bundles on X:

(1) There exists a canonical isomorphism $H^0(X, \mathcal{O}(d)) \cong B^{\varphi=p^d}$ for every $d \in \mathbb{Z}$.
(2) The cohomology group $H^1(X, \mathcal{O}(d))$ vanishes for every $d \geq 0$.

PROOF. Take a nonzero element $t \in B^{\varphi=p}$. By Corollary 3.1.9 there exists a closed point x on X induced by t. Let us write $U := X \setminus \{ x \}$. Then we have an identification $U \cong \mathrm{Spec}\,(B[1/t]^{\varphi=1})$.

For every $d \in \mathbb{Z}$, the multiplication by t on P yields an injective map of P-graded modules $P(d) \hookrightarrow P(d+1)$ by Corollary 2.1.17, and in turn gives rise to an injective sheaf morphism $\mathcal{O}(d) \hookrightarrow \mathcal{O}(d+1)$. In addition, Proposition 3.2.5 implies that each $\mathcal{O}(d)$ is isomorphic to the line bundle that arises from the Weil divisor dx. We then find that $\varinjlim \mathcal{O}(d)$ is naturally isomorphic to the pushforward of $\mathcal{O}_U$ by the embedding $U \hookrightarrow X$, and in turn obtain identifications

$$H^0\left(X, \varinjlim \mathcal{O}(d)\right) \cong H^0(U, \mathcal{O}_U) \cong B[1/t]^{\varphi=1}, \tag{3.7}$$
$$H^1\left(X, \varinjlim \mathcal{O}(d)\right) \cong H^1(U, \mathcal{O}_U) = 0. \tag{3.8}$$

Let us now prove the statement (1). For every $d \in \mathbb{Z}$, we denote by α_d the saturation map of $P(d)$. We wish to show that each α_d is an isomorphism. Proposition 3.2.8 implies that the sequence (α_d) gives rise to a map

$$B[1/t]^{\varphi=1} \cong \varinjlim B^{\varphi=p^d} \longrightarrow \varinjlim H^0(X, \mathcal{O}(d)) \cong H^0\left(X, \varinjlim \mathcal{O}(d)\right),$$

which is easily seen to coincide with the isomorphism (3.7). Moreover, Proposition 3.2.8 and the snake lemma together yield isomorphisms

$$\ker(\alpha_d) \simeq \ker(\alpha_{d+1}) \qquad \text{and} \qquad \mathrm{coker}(\alpha_d) \simeq \mathrm{coker}(\alpha_{d+1}) \qquad \text{for all } d \geq 0.$$

Therefore we deduce that α_d is an isomorphism for every $d \geq 0$. In particular, we have $H^0(X, \mathcal{O}_X) \cong B^{\varphi=1} \cong \mathbb{Q}_p$ where the second isomorphism is given by Proposition 3.1.6. Then for every $d < 0$, we find that there exists no nonzero element element of $H^0(X, \mathcal{O}_X)$ which vanishes to order $-d$ at x, and consequently obtain $H^0(X, \mathcal{O}(d)) = 0$. We thus deduce by Proposition 2.1.15 that α_d is an isomorphism for every $d < 0$ as well.

It remains to establish the statement (2). For every $n \geq 0$, the last statement of Proposition 3.2.8 implies that the cohomology of $\mathcal{O}(d+1)/t\mathcal{O}(d)$ vanishes in degree 1. Hence for every $d \geq 0$ we have a long exact sequence

$$H^0(X, \mathcal{O}(d+1)) \longrightarrow H^0(X, \mathcal{O}(d+1)/t\mathcal{O}(d)) \longrightarrow H^1(X, \mathcal{O}(d)) \longrightarrow H^1(X, \mathcal{O}(d+1)) \longrightarrow 0,$$

which in turn yields an isomorphism $H^1(X, \mathcal{O}(d)) \simeq H^1(X, \mathcal{O}(d+1))$ as the first arrow is surjective by Proposition 3.2.8. The desired assertion now follows by (3.8). □

Remark. Theorem 3.2.9 provides analogues of the following facts about the complex projective line $\mathbb{P}^1_{\mathbb{C}} = \mathrm{Proj}\,(\mathbb{C}[z_1, z_2])$:

(1) For every $d \in \mathbb{Z}$, the cohomology group $H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d))$ is naturally isomorphic to the group of degree d homogeneous polynomials in $\mathbb{C}[z_1, z_2]$.
(2) For every $d \geq 0$, the cohomology group $H^1(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d))$ vanishes.

However, it is known that $H^1(X, \mathcal{O}(-1))$ does not vanish while $H^1(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1))$ vanishes.

3.3. Harder-Narasimhan filtration

In this subsection, we review the Harder-Narasimhan formalism for vector bundles on a complete algebraic curve.

Definition 3.3.1. A *complete algebraic curve* is a scheme Z with the following properties:

(i) Z is connected, separated, noetherian and regular of dimension 1.
(ii) The Picard group $\mathrm{Pic}(Z)$ admits a homomorphism $\deg_Z : \mathrm{Pic}(Z) \longrightarrow \mathbb{Z}$, called a *degree map*, which takes a positive value on every line bundle that arises from a nonzero effective Weil divisor on Z.

Example 3.3.2. Below are two important examples of complete algebraic curves.

(1) Every regular proper curve over a field is a complete algebraic curve by a general fact as stated in [**Sta**, Tag 0AYY].
(2) The Fargues-Fontaine curve is a complete algebraic curve by Theorem 2.4.8 and Proposition 3.2.5.

For the rest of this subsection, we fix a complete algebraic curve Z with a degree map $\deg_Z$ on the Picard group $\mathrm{Pic}(Z)$. Our first goal in this subsection is to study the notion of degree and slope for vector bundles on Z.

Definition 3.3.3. Let $\mathcal{V}$ be a vector bundle on Z.

(1) We write $\mathrm{rk}(\mathcal{V})$ for the rank of $\mathcal{V}$, and define the *degree* of $\mathcal{V}$ by

$$\deg(\mathcal{V}) := \deg_Z \left(\wedge^{\mathrm{rk}(\mathcal{V})}(\mathcal{V}) \right).$$

(2) If $\mathcal{V}$ is not zero, we define its *slope* by

$$\mu(\mathcal{V}) := \frac{\deg(\mathcal{V})}{\mathrm{rk}(\mathcal{V})}.$$

(3) We denote by $\mathcal{V}^\vee$ the dual bundle of $\mathcal{V}$.

Proposition 3.3.4. Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be vector bundles on Z. Assume that there exits a short exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0.$$

(1) We have identities

$$\mathrm{rk}(\mathcal{V}) = \mathrm{rk}(\mathcal{U}) + \mathrm{rk}(\mathcal{W}) \qquad \text{and} \qquad \deg(\mathcal{V}) = \deg(\mathcal{U}) + \deg(\mathcal{W}).$$

(2) If $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are all nonzero, then we have

$$\min\left(\mu(\mathcal{U}), \mu(\mathcal{W})\right) \leq \mu(\mathcal{V}) \leq \max\left(\mu(\mathcal{U}), \mu(\mathcal{W})\right)$$

with equality if and only if $\mu(\mathcal{U})$ and $\mu(\mathcal{W})$ are equal.

Proof. The first identity in the statement (1) is evident, whereas the second identity in the statement (1) follows from a general fact as stated in [**Sta**, Tag 0B38]. It remains to prove the the statement (2). Let us now assume that $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are all nonzero. By the statement (1) we have

$$\mu(\mathcal{V}) = \frac{\deg(\mathcal{V})}{\mathrm{rk}(\mathcal{V})} = \frac{\deg(\mathcal{U}) + \deg(\mathcal{W})}{\mathrm{rk}(\mathcal{U}) + \mathrm{rk}(\mathcal{W})}.$$

If $\mu(\mathcal{U})$ and $\mu(\mathcal{W})$ are not equal, then $\mu(\mathcal{V})$ must lie between $\mu(\mathcal{U})$ and $\mu(\mathcal{W})$. Otherwise, we find $\mu(\mathcal{U}) = \mu(\mathcal{V}) = \mu(\mathcal{W})$. ∎

LEMMA 3.3.5. Let M and N be free modules over a ring R of rank r and r'. There exists a canonical isomorphism

$$\wedge^{rr'}(M \otimes_R N) \cong \wedge^r(M)^{\otimes r'} \otimes_R \wedge^{r'}(N)^{\otimes r}.$$

PROOF. Let us choose bases (m_i) and (n_j) for M and N, respectively. We have an isomorphism of rank 1 free R-modules

$$\wedge^{rr'}(M \otimes_R N) \simeq \wedge^r(M)^{\otimes r'} \otimes_R \wedge^{r'}(N)^{\otimes r} \tag{3.9}$$

which maps $\bigwedge(m_i \otimes n_j)$ to $(\bigwedge m_i)^{\otimes r'} \otimes (\bigwedge n_j)^{\otimes r}$. It suffices to show that this map does not depend on the choices of (m_i) and (n_j). Take an invertible $r \times r$ matrix $\alpha = (\alpha_{h,i})$ over R. Then we have

$$\bigwedge\left(\sum \alpha_{h,i} m_i \otimes n_j\right) = \det(\alpha)^{r'} \bigwedge(m_i \otimes n_j),$$
$$\left(\bigwedge\left(\sum \alpha_{h,i} m_i\right)\right)^{\otimes r'} \otimes \left(\bigwedge n_j\right)^{\otimes r} = \det(\alpha)^{r'} \left(\bigwedge m_i\right)^{\otimes r'} \otimes \left(\bigwedge n_j\right)^{\otimes r}.$$

Hence $\bigwedge(\sum \alpha_{h,i} m_i \otimes n_j)$ maps to $(\bigwedge(\sum \alpha_{h,i} m_i))^{\otimes r'} \otimes (\bigwedge n_j)^{\otimes r}$ under (3.9). It follows that the map (3.9) does not depend on the choice of (m_i). By symmetry, the map (3.9) does not depend on the choice of (n_j) either. Therefore we deduce the desired assertion. □

PROPOSITION 3.3.6. Let $\mathcal{V}$ and $\mathcal{W}$ be nonzero vector bundles on Z. Then we have

$$\deg(\mathcal{V} \otimes_{\mathcal{O}_Z} \mathcal{W}) = \deg(\mathcal{V})\mathrm{rk}(\mathcal{W}) + \deg(\mathcal{W})\mathrm{rk}(\mathcal{V}) \quad \text{and} \quad \mu(\mathcal{V} \otimes_{\mathcal{O}_Z} \mathcal{W}) = \mu(\mathcal{V}) + \mu(\mathcal{W}).$$

PROOF. Since we have $\mathrm{rk}(\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{W}) = \mathrm{rk}(\mathcal{V})\mathrm{rk}(\mathcal{W})$, the first identity is straightforward to verify by Lemma 3.3.5. The second identity then immediately follows. □

LEMMA 3.3.7. The cohomology group $H^0(Z, \mathcal{O}_Z)$ is a field.

PROOF. Let $K(Z)$ denote the function field of Z, and take an arbitrary element $f \in K(Z)^\times$. Then f yields a global section of $\mathcal{O}_Z$ if and only if the associated Weil divisor $\mathrm{Div}(f)$ on Z is effective. Since every principal divisor on Z induces a line bundle of degree 0, the Weil divisor $\mathrm{Div}(f)$ is effective if and only if it is the zero divisor. We thus find

$$H^0(Z, \mathcal{O}_Z) \backslash \{ 0 \} = \left\{ f \in K(Z)^\times : \mathrm{Div}(f) = 0 \right\},$$

and consequently deduce that $H^0(Z, \mathcal{O}_Z)$ is a subfield of $K(Z)$. □

LEMMA 3.3.8. Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on Z.

(1) If we have $\deg(\mathcal{L}) > \deg(\mathcal{M})$, there is no nonzero $\mathcal{O}_Z$-module map from $\mathcal{L}$ to $\mathcal{M}$.
(2) If we have $\deg(\mathcal{L}) = \deg(\mathcal{M})$, every nonzero $\mathcal{O}_Z$-module map from $\mathcal{L}$ to $\mathcal{M}$ is an isomorphism.

PROOF. Assume that there exists a nonzero $\mathcal{O}_Z$-module map $s : \mathcal{L} \longrightarrow \mathcal{M}$. Then s induces a nonzero global section on $\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}$ via the identification

$$\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{L}, \mathcal{M}) \cong H^0(Z, \mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}). \tag{3.10}$$

Hence $\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}$ arises from an effective Weil divisor D on Z by a general fact as stated in [**Sta**, Tag 01X0]. We then find

$$\deg(\mathcal{M}) - \deg(\mathcal{L}) = \deg(\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}) \geq 0, \tag{3.11}$$

and consequently deduce the first statement.

Let us now assume $\deg(\mathcal{L}) = \deg(\mathcal{M})$. By (3.11) we have $\deg(\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}) = 0$, which means that the effective Weil D must be zero. It follows that $\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}$ is trivial, which in turn implies by (3.10) and Lemma 3.3.7 that s is an isomorphism. □

Proposition 3.3.9. A coherent $\mathcal{O}_Z$-module is a vector bundle if and only if it is torsion free.

Proof. Since Z is integral and regular by construction, the assertion follows from a general fact as stated in [**Sta**, Tag 0CC4]. □

Proposition 3.3.10. Let $\mathcal{V}$ be a vector bundle on Z, and let $\mathcal{W}$ be a coherent subsheaf of $\mathcal{V}$.

(1) $\mathcal{W}$ is a vector bundle on Z.
(2) $\mathcal{W}$ is contained in a subbundle $\widetilde{\mathcal{W}}$ of $\mathcal{V}$ with $\mathrm{rk}(\mathcal{W}) = \mathrm{rk}(\widetilde{\mathcal{W}})$ and $\deg(\mathcal{W}) \leq \deg(\widetilde{\mathcal{W}})$.

Proof. Since $\mathcal{W}$ is evidently torsion free, the first statement follows from Proposition 3.3.9. Hence it remains to verify the second statement. We may assume $\mathcal{W} \neq 0$, as otherwise the assertion would be obvious. Let $\mathcal{T}$ denote the torsion subsheaf of the quotient $\mathcal{V}/\mathcal{W}$. Take $\widetilde{\mathcal{W}}$ to be the preimage of $\mathcal{T}$ under the surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{W}$. Then $\widetilde{\mathcal{W}}$ is a torsion free subsheaf of $\mathcal{V}$ with a torsion free quotient, and thus is a subbundle of $\mathcal{V}$ by Proposition 3.3.9. In addition, we have $\mathcal{W} \subseteq \widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{W}}/\mathcal{W} \simeq \mathcal{T}$ by construction, and consequently find $\mathrm{rk}(\widetilde{\mathcal{W}}) = \mathrm{rk}(\mathcal{W})$ as $\mathcal{T}$ has rank 0 for being a torsion sheaf. We also have a nonzero $\mathcal{O}_Z$-module map $\wedge^{\mathrm{rk}(\mathcal{W})}\mathcal{W} \longrightarrow \wedge^{\mathrm{rk}(\widetilde{\mathcal{W}})}\widetilde{\mathcal{W}}$ induced by the embedding $\mathcal{W} \hookrightarrow \widetilde{\mathcal{W}}$, and in turn obtain $\deg(\mathcal{W}) \leq \deg(\widetilde{\mathcal{W}})$ by Lemma 3.3.8. □

Remark. The subbundle $\widetilde{\mathcal{W}}$ of $\mathcal{V}$ that we constructed above is often referred to as the *saturation* of $\mathcal{W}$ in $\mathcal{V}$.

Proposition 3.3.11. Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on Z of equal rank and degree. Assume that $\mathcal{W}$ is a coherent subsheaf of $\mathcal{V}$. Then we have $\mathcal{V} = \mathcal{W}$.

Proof. The embedding $\mathcal{W} \hookrightarrow \mathcal{V}$ induces a nonzero map $\wedge^{\mathrm{rk}(\mathcal{W})}(\mathcal{W}) \longrightarrow \wedge^{\mathrm{rk}(\mathcal{V})}(\mathcal{V})$, which is forced to be an isomorphism by Lemma 3.3.8. Hence at each point in Z the embedding $\mathcal{W} \hookrightarrow \mathcal{V}$ yields an isomorphism on the stalks for having an invertible determinant. It follows that the embedding $\mathcal{W} \hookrightarrow \mathcal{V}$ is an isomorphism. □

Proposition 3.3.12. Given a vector bundle $\mathcal{V}$ on Z, there is an integer $d_{\mathcal{V}}$ with $\deg(\mathcal{W}) \leq d_{\mathcal{V}}$ for every subbundle $\mathcal{W}$ of $\mathcal{V}$.

Proof. If $\mathcal{V}$ is the zero bundle, the assertion is trivial. Let us now proceed by induction on $\mathrm{rk}(\mathcal{V})$. We may assume that there exists a nonzero proper subbundle $\mathcal{U}$ of $\mathcal{V}$, as otherwise the assertion would be obvious. Consider an arbitrary subbundle $\mathcal{W}$ of $\mathcal{V}$. Let us set $\mathcal{P} := \mathcal{W} \cap \mathcal{U}$ and denote by $\mathcal{Q}$ the image of $\mathcal{W}$ under the natural surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{U}$. Proposition 3.3.10 and the induction hypothesis together imply that $\mathcal{P}$ and $\mathcal{Q}$ are vector bundles on Z with

$$\deg(\mathcal{P}) \leq d_{\mathcal{U}} \qquad \text{and} \qquad \deg(\mathcal{Q}) \leq d_{\mathcal{V}/\mathcal{U}}$$

for some integers $d_{\mathcal{U}}$ and $d_{\mathcal{V}/\mathcal{U}}$ that do not depend on $\mathcal{W}$. In addition, we have a short exact sequence

$$0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{W} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Therefore we obtain

$$\deg(\mathcal{W}) = \deg(\mathcal{P}) + \deg(\mathcal{Q}) \leq d_{\mathcal{U}} + d_{\mathcal{V}/\mathcal{U}}$$

where the first identity follows from Proposition 3.3.4. □

Remark. On the other hand, if $\mathcal{V}$ is not a line bundle on Z, we don't necessarily have an integer $d'_{\mathcal{V}}$ with $\deg(\mathcal{W}) \geq d'_{\mathcal{V}}$ for every subbundle $\mathcal{W}$ of $\mathcal{V}$. In fact, in the context of the complex projective line or the Fargues-Fontaine curve, it is known that such an integer $d'_{\mathcal{V}}$ never exists if $\mathcal{V}$ is not a line bundle.

We now introduce and study two important classes of vector bundles on Z.

Definition 3.3.13. Let $\mathcal{V}$ be a nonzero vector bundle on Z.

(1) We say that $\mathcal{V}$ is *semistable* if we have $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$ for every nonzero subbundle $\mathcal{W}$ of $\mathcal{V}$.

(2) We say that $\mathcal{V}$ is *stable* if we have $\mu(\mathcal{W}) < \mu(\mathcal{V})$ for every nonzero proper subbundle $\mathcal{W}$ of $\mathcal{V}$.

Remark. Here we don't speak of semistability for the zero bundle, although some authors say that the zero bundle is semistable of every slope.

Example 3.3.14. Every line bundle on Z is stable; indeed, a line bundle on Z has no nonzero proper subbundles as easily seen by Proposition 3.3.4.

PROPOSITION 3.3.15. Let $\mathcal{V}$ be a semistable vector bundle on Z. Every nonzero coherent subsheaf $\mathcal{W}$ of $\mathcal{V}$ is a vector bundle on Z with $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$.

PROOF. Proposition 3.3.10 implies that $\mathcal{W}$ is a vector bundle on Z, contained in some subbundle $\widetilde{\mathcal{W}}$ of $\mathcal{V}$ with $\mu(\mathcal{W}) \leq \mu(\widetilde{\mathcal{W}})$. We then find $\mu(\widetilde{\mathcal{W}}) \leq \mu(\mathcal{V})$ by the semistability of $\mathcal{V}$, and consequently obtain the desired assertion. □

PROPOSITION 3.3.16. Let $\mathcal{V}$ and $\mathcal{W}$ be semistable vector bundles on Z with $\mu(\mathcal{V}) > \mu(\mathcal{W})$. Then we have $\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{V}, \mathcal{W}) = 0$.

PROOF. Suppose for contradiction that there is a nonzero $\mathcal{O}_Z$-module map $f : \mathcal{V} \longrightarrow \mathcal{W}$. Let $\mathcal{Q}$ denote the image of f. Proposition 3.3.15 implies that $\mathcal{Q}$ is a vector bundle on Z with

$$\mu(\mathcal{Q}) \leq \mu(\mathcal{W}) < \mu(\mathcal{V}). \tag{3.12}$$

Let us now consider the short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow \mathcal{V} \xrightarrow{\ f\ } \mathcal{Q} \longrightarrow 0.$$

We have $\ker(f) \neq 0$ as $\mathcal{Q}$ and $\mathcal{V}$ are not isomorphic by (3.12). We thus obtain $\mu(\ker(f)) \leq \mu(\mathcal{V})$ by the semistability of $\mathcal{V}$ and consequently find $\mu(\mathcal{Q}) \geq \mu(\mathcal{V})$ by Proposition 3.3.4, thereby deducing a desired contradiction by (3.12). □

Remark. The converse of Proposition 3.3.16 does not hold in general. For example, if the Picard group of Z is not isomorphic to $\mathbb{Z}$, we get a nontrivial degree 0 line bundle $\mathcal{L}$ on Z and find $\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{O}_Z, \mathcal{L}) = 0$ by Lemma 3.3.8. On the other hand, if Z is taken to be the complex projective line or the Fargues-Fontaine curve, then the converse of Proposition 3.3.16 is known to hold.

PROPOSITION 3.3.17. Let $\mathcal{V}$ be a vector bundle on Z such that $\mathcal{V}^{\otimes n}$ is semistable for some $n > 0$. Then $\mathcal{V}$ is semistable.

PROOF. Consider an arbitrary nonzero subbundle $\mathcal{W}$ of $\mathcal{V}$. We may regard $\mathcal{W}^{\otimes n}$ as a subsheaf of $\mathcal{V}^{\otimes n}$. Then we have $\mu(\mathcal{W}^{\otimes n}) \leq \mu(\mathcal{V}^{\otimes n})$ by Proposition 3.3.15, and in turn find

$$\mu(\mathcal{W}) = \mu(\mathcal{W}^{\otimes n})/n \leq \mu(\mathcal{V}^{\otimes n})/n = \mu(\mathcal{V})$$

by Proposition 3.3.6. □

Remark. It is natural to ask if the tensor product of two arbitrary semistable vector bundles on Z is necessarily semistable. If Z is a regular proper curve over a field of characteristic 0, this is known to be true by the work of Narasimhan-Seshadri [**NS65**]. In addition, we will see in Corollary 3.5.2 that this is true in the context of the Fargues-Fontaine curve. However, this is false if Z is defined over a field of characteristic p, as shown by Gieseker [**Gie73**].

PROPOSITION 3.3.18. Let $\mathcal{V}$ and $\mathcal{W}$ be semistable vector bundles on Z of slope λ.

(1) Every extension of $\mathcal{W}$ by $\mathcal{V}$ is a semistable vector bundle on Z of slope λ.
(2) For every $f \in \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{V}, \mathcal{W})$, both $\ker(f)$ and $\mathrm{coker}(f)$ are either trivial or semistable vector bundles on Z of slope λ.

PROOF. Let $\mathcal{E}$ be a vector bundle on X which fits into a short exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{E} \longrightarrow \mathcal{W} \longrightarrow 0.$$

By Proposition 3.3.4 we find $\mu(\mathcal{E}) = \lambda$. Take an arbitrary subbundle $\mathcal{F}$ of $\mathcal{E}$, and denote by $\mathcal{F}'$ its image under the map $\mathcal{E} \twoheadrightarrow \mathcal{W}$. By construction we have a short exact sequence

$$0 \longrightarrow \mathcal{V} \cap \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

In addition, Proposition 3.3.15 implies that $\mathcal{V} \cap \mathcal{F}$ and $\mathcal{F}'$ are vector bundles on Z with

$$\mu(\mathcal{V} \cap \mathcal{F}) \leq \mu(\mathcal{V}) = \lambda \qquad \text{and} \qquad \mu(\mathcal{F}') \leq \mu(\mathcal{W}) = \lambda.$$

We then find $\mu(\mathcal{F}) \leq \lambda = \mu(\mathcal{E})$ by Proposition 3.3.4, thereby deducing the statement (1).

It remains prove the statement (2). The assertion is trivial for $f = 0$. We henceforth assume $f \neq 0$, and denote by $\mathcal{Q}$ the image of f. Then we have a short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow \mathcal{V} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

Moreover, Proposition 3.3.15 implies that $\ker(f)$ and $\mathcal{Q}$ are vector bundles on Z with

$$\deg(\ker(f)) \leq \mu(\mathcal{V}) \cdot \mathrm{rk}(\ker(f)) = \lambda \cdot \mathrm{rk}(\ker(f)) \qquad \text{and} \qquad \mu(\mathcal{Q}) \leq \mu(\mathcal{W}) = \lambda.$$

Hence by Proposition 3.3.4 we find

$$\deg(\ker(f)) = \lambda \cdot \mathrm{rk}(\ker(f)) \qquad \text{and} \qquad \mu(\mathcal{Q}) = \lambda.$$

Since every subbundle of $\ker(f)$ is a coherent subsheaf of $\mathcal{V}$, the first identity and Proposition 3.3.15 together imply that $\ker(f)$ is either zero or semistable of slope λ.

Meanwhile, Proposition 3.3.10 implies that $\mathcal{Q}$ is contained in a subbundle $\widetilde{\mathcal{Q}}$ of $\mathcal{W}$ with

$$\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\widetilde{\mathcal{Q}}) \qquad \text{and} \qquad \deg(\mathcal{Q}) \leq \deg(\widetilde{\mathcal{Q}}). \tag{3.13}$$

Then by the semistability of $\mathcal{V}$ we obtain

$$\lambda = \mu(\mathcal{Q}) \leq \mu(\widetilde{\mathcal{Q}}) \leq \mu(\mathcal{W}) = \lambda,$$

and consequently find that the inequality in (3.13) is indeed an equality. Hence Proposition 3.3.11 yields $\mathcal{Q} = \widetilde{\mathcal{Q}}$, which in particular means that $\mathcal{Q}$ is a subbundle of $\mathcal{W}$.

Let us now assume that $\mathrm{coker}(f)$ is not zero. Since we have a short exact sequence

$$0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{W} \longrightarrow \mathrm{coker}(f) \longrightarrow 0,$$

our discussion in the preceding paragraph and Proposition 3.3.4 together imply that $\mathrm{coker}(f)$ is a vector bundle on Z with $\mu(\mathrm{coker}(f)) = \lambda$. We wish to show that $\mathrm{coker}(f)$ is semistable. Take an arbitrary subbundle $\mathcal{R}$ of $\mathrm{coker}(f)$, and denote by $\mathcal{R}'$ its preimage under the map $\mathcal{W} \twoheadrightarrow \mathrm{coker}(f)$. Then we have a short exact sequence

$$0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{R}' \longrightarrow \mathcal{R} \longrightarrow 0.$$

In addition, Proposition 3.3.15 implies that $\mathcal{R}'$ is a vector bundle on Z with

$$\mu(\mathcal{R}') \leq \mu(\mathcal{W}) = \lambda = \mu(\mathcal{Q}).$$

Hence we find $\mu(\mathcal{R}) \leq \mu(\mathcal{Q}) = \lambda = \mu(\mathrm{coker}(f))$ by Proposition 3.3.4, and consequently deduce that $\mathrm{coker}(f)$ is semistable as desired. $\square$

Our final goal in this subsection is to show that every vector bundle on Z admits a unique filtration whose successive quotients are semistable vector bundles with strictly increasing slopes.

Definition 3.3.19. Let $\mathcal{V}$ be a vector bundle on Z. A *Harder-Narasimhan filtration* of $\mathcal{V}$ is a filtration by subbundles

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}$$

such that the successive quotients $\mathcal{V}_1/\mathcal{V}_0, \cdots, \mathcal{V}_n/\mathcal{V}_{n-1}$ are semistable vector bundles on Z with $\mu(\mathcal{V}_1/\mathcal{V}_0) > \cdots > \mu(\mathcal{V}_n/\mathcal{V}_{n-1})$.

Lemma 3.3.20. Given a nonzero vector bundle $\mathcal{V}$ on Z, there exists a semistable subbundle $\mathcal{V}_1$ of $\mathcal{V}$ with $\mu(\mathcal{V}_1) \geq \mu(\mathcal{V})$ and $\mu(\mathcal{V}_1) > \mu(\mathcal{U})$ for every nonzero subbundle $\mathcal{U}$ of $\mathcal{V}/\mathcal{V}_1$.

Proof. For an arbitrary nonzero subbundle $\mathcal{W}$ of $\mathcal{V}$, we have $0 < \mathrm{rk}(\mathcal{W}) \leq \mathrm{rk}(\mathcal{V})$ and $\deg(\mathcal{W}) \leq d_{\mathcal{V}}$ for some fixed integer $d_{\mathcal{V}}$ given by Proposition 3.3.12. This implies that the set

$$S := \{ \, q \in \mathbb{Q} : q = \mu(\mathcal{W}) \text{ for some nonzero subbundle } \mathcal{W} \text{ of } \mathcal{V} \, \}$$

is discrete and bounded above. In particular, the set S contains the largest element λ.

Let us take $\mathcal{V}_1$ to be a maximal subbundle of $\mathcal{V}$ with $\mu(\mathcal{V}_1) = \lambda$. By construction we have $\mu(\mathcal{V}_1) \geq \mu(\mathcal{V})$. Moreover, since every subbundle of $\mathcal{V}_1$ is a coherent subsheaf of $\mathcal{V}$, Proposition 3.3.10 and the maximality of λ together imply that $\mathcal{V}_1$ is semistable. Let us now consider an arbitrary nonzero subbundle $\mathcal{U}$ of $\mathcal{V}/\mathcal{V}_1$, and denote by $\widetilde{\mathcal{U}}$ its preimage under the natural surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{V}_1$. Then we have a short exact sequence

$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \widetilde{\mathcal{U}} \longrightarrow \mathcal{U} \longrightarrow 0.$$

In addition, the maximality of λ and $\mathcal{V}_1$ implies $\mu(\widetilde{\mathcal{U}}) < \lambda = \mu(\mathcal{V}_1)$. Therefore we find $\mu(\mathcal{U}) < \mu(\mathcal{V}_1)$ by Proposition 3.3.4, thereby completing the proof. □

Remark. Our proof above relies on the fact that the group $\mathbb{Z}$ is discrete. However, as noted in [**Ked19**, Lemma 3.4.10], it is not hard to prove Lemma 3.3.20 without using the discreteness of $\mathbb{Z}$. As a consequence, we can extend all of our discussion in this subsection to some other contexts where the degree of a vector bundle takes a value in a nondiscrete group such as $\mathbb{Z}[1/p]$. We refer the curious readers to [**Ked19**, Example 3.5.7] for a discussion of such an example.

Lemma 3.3.21. Let $\mathcal{V}$ be a nonzero vector bundle on Z. Assume that $\mathcal{V}$ admits a Harder-Narasimhan filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}.$$

For every semistable vector bundle $\mathcal{W}$ on Z with $\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{W}, \mathcal{V}) \neq 0$, we have $\mu(\mathcal{W}) \leq \mu(\mathcal{V}_1)$.

Proof. Take a nonzero $\mathcal{O}_Z$-module map $f : \mathcal{W} \longrightarrow \mathcal{V}$, and denote its image by $\mathcal{Q}$. Since $\mathcal{Q}$ is a nonzero coherent subsheaf of $\mathcal{V}$, there exists the smallest integer $i \geq 1$ with $\mathcal{Q} \subseteq \mathcal{V}_i$. Then we find that f induces a nonzero $\mathcal{O}_Z$-module map $\mathcal{W} \xrightarrow{f} \mathcal{V}_i \twoheadrightarrow \mathcal{V}_i/\mathcal{V}_{i-1}$, and consequently obtain

$$\mu(\mathcal{W}) \leq \mu(\mathcal{V}_i/\mathcal{V}_{i-1}) \leq \mu(\mathcal{V}_1)$$

where the first inequality follows by Proposition 3.3.16. □

Remark. Lemma 3.3.21 does not hold without the semistability assumption on $\mathcal{W}$. For example, if we take $\mathcal{W} := \mathcal{V} \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on Z with $\mu(\mathcal{L}) > \mu(\mathcal{V})$, we find $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{W}, \mathcal{V}) \neq 0$ and $\mu(\mathcal{W}) > \mu(\mathcal{V})$.

THEOREM 3.3.22 (Harder-Narasimhan [**HN75**]). Every vector bundle $\mathcal{V}$ on Z admits a unique Harder-Narasimhan filtration.

PROOF. Let us proceed by induction on $\mathrm{rk}(\mathcal{V})$. If $\mathcal{V}$ is the zero bundle, the assertion is trivial. We henceforth assume that $\mathcal{V}$ is not zero.

We first assert that $\mathcal{V}$ admits a Harder-Narasimhan filtration. Lemma 3.3.20 yields a semistable subbundle $\mathcal{V}_1$ of $\mathcal{V}$ with $\mu(\mathcal{V}_1) > \mu(\mathcal{U})$ for every nonzero subbundle $\mathcal{U}$ of $\mathcal{V}/\mathcal{V}_1$. By the induction hypothesis, the vector bundle $\mathcal{V}/\mathcal{V}_1$ on Z admits a Harder-Narasimhan filtration

$$0 = \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_n = \mathcal{V}/\mathcal{V}_1. \tag{3.14}$$

For each $i = 2, \cdots, n$, let us set $\mathcal{V}_i$ to be the preimage of $\mathcal{U}_i$ under the natural surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{V}_1$. Then we find

$$\mathcal{V}_i/\mathcal{V}_{i-1} \cong \mathcal{U}_i/\mathcal{U}_{i-1} \qquad \text{for each } i = 2, \cdots, n.$$

Moreover, by construction we have $\mu(\mathcal{V}_1) > \mu(\mathcal{U}_2)$ whenever the filtration (3.14) is not trivial. Therefore $\mathcal{V}$ admits a Harder-Narasimhan filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}. \tag{3.15}$$

It remains to show that (3.15) is a unique Harder-Narasimhan filtration of $\mathcal{V}$. Assume that $\mathcal{V}$ admits another Harder-Narasimhan filtration

$$0 = \mathcal{W}_0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_l = \mathcal{V}. \tag{3.16}$$

Since $\mathcal{W}_1$ is a nonzero subbundle of $\mathcal{V}$, Lemma 3.3.21 yields $\mu(\mathcal{W}_1) \leq \mu(\mathcal{V}_1)$. Then by symmetry we obtain $\mu(\mathcal{V}_1) \leq \mu(\mathcal{W}_1)$, and thus find $\mu(\mathcal{V}_1) = \mu(\mathcal{W}_1)$. Now we have

$$\mu(\mathcal{W}_1) = \mu(\mathcal{V}_1) > \mu(\mathcal{V}_2/\mathcal{V}_1) = \mu(\mathcal{U}_2/\mathcal{U}_1)$$

unless the filtration (3.14) is trivial. It follows by Lemma 3.3.21 that $\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{W}_1, \mathcal{V}/\mathcal{V}_1)$ vanishes. We then find $\mathcal{W}_1 \subseteq \mathcal{V}_1$ by observing that the natural map $\mathcal{W}_1 \hookrightarrow \mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{V}_1$ must be zero. By symmetry we also obtain $\mathcal{V}_1 \subseteq \mathcal{W}_1$, and consequently deduce that $\mathcal{V}_1$ and $\mathcal{W}_1$ are equal. The filtration (3.16) then induces a Harder-Narasimhan filtration

$$0 = \mathcal{W}_1/\mathcal{V}_1 \subset \cdots \mathcal{W}_l/\mathcal{V}_1 = \mathcal{V}/\mathcal{V}_1, \tag{3.17}$$

which must coincide with the filtration (3.14) by the induction hypothesis. Since each $\mathcal{W}_i$ is the preimage of $\mathcal{W}_i/\mathcal{V}_1$ under the natural surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{V}_1$, we deduce that the filtrations (3.15) and (3.16) coincide. □

Remark. A careful examination of our proof shows that Theorem 3.3.22 is a formal consequence of Proposition 3.3.4 and Proposition 3.3.10. In other words, Theorem 3.3.22 readily extends to any exact category $\mathscr{C}$ equipped with assignments $\mathrm{rk}_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathbb{Z}_{\geq 0}$ and $\deg_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathbb{Z}$ that satisfy the following properties:

(i) Both $\mathrm{rk}_{\mathscr{C}}$ and $\deg_{\mathscr{C}}$ are additive on short exact sequences.

(ii) Every monomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathscr{C}$ factors through some admissible monomorphism $\widetilde{f} : \widetilde{\mathcal{A}} \longrightarrow \mathcal{B}$ with $\mathrm{rk}_{\mathscr{C}}(\mathcal{A}) = \mathrm{rk}_{\mathscr{C}}(\widetilde{\mathcal{A}})$ and $\deg_{\mathscr{C}}(\mathcal{A}) \leq \deg_{\mathscr{C}}(\widetilde{\mathcal{A}})$.

Such a category is called a *slope category*.

We will see that the category of vector bundles on the Fargues-Fontaine curve is closely related to two other slope categories, namely the category of isocrystals and the category of filtered isocrystals. This fact will be crucial for studying the essential image of the crystalline functor in §4.2.

3.4. Semistable bundles and unramified covers

In this subsection, we construct semistable vector bundles on the Fargues-Fontaine curve by studying its unramified covers.

Definition 3.4.1. Let h be a positive integer.

(1) We denote by E_h the degree r unramified extension of $\mathbb{Q}_p$, and define the *degree h unramified cover* of X to be the natural map
$$\pi_h : X \times_{\mathrm{Spec}\,(\mathbb{Q}_p)} \mathrm{Spec}\,(E_h) \longrightarrow X.$$
(2) We write $X_h := X \times_{\mathrm{Spec}\,(\mathbb{Q}_p)} \mathrm{Spec}\,(E_h)$ and $P_h := \bigoplus_{n\geq 0} B^{\varphi^h = p^n}$.

Lemma 3.4.2. Let r and n be integers with $r > 0$. Given a positive integer h and a nonzero homogeneous element $f \in P$, we have a canonical isomorphism
$$B[1/f]^{\varphi^r = p^n} \otimes_{\mathbb{Q}_p} E_h \cong B[1/f]^{\varphi^{rh} = p^{nh}}.$$

Proof. The group $\mathrm{Gal}(E_h/\mathbb{Q}_p)$ is cyclic of order h, and admits a canonical generator γ which lifts the p-th power map on $\mathbb{F}_{p^h}$. Moreover, for every $n \in \mathbb{Z}$ there exists an action of $\mathrm{Gal}(E_h/\mathbb{Q}_p)$ on $B[1/f]^{\varphi^{rh}=p^{nh}}$ such that γ acts via $p^{-n}\varphi^r$. We thus find
$$B[1/f]^{\varphi^r = p^n} = \left(B[1/f]^{\varphi^{rh}=p^{nh}}\right)^{\mathrm{Gal}(E_h/\mathbb{Q}_p)},$$
and consequently deduce the desired isomorphism by the Galois descent for vector spaces. □

Proposition 3.4.3. For every positive integer h, we have a canonical isomorphism
$$X_h \cong \mathrm{Proj}\,(P_h).$$

Proof. By Lemma 3.4.2 we have $B^{\varphi = p^n} \otimes_{\mathbb{Q}_p} E_h \cong B^{\varphi^h = p^{nh}}$ for every $n \in \mathbb{Z}$, and consequently obtain a natural isomorphism
$$X_h \cong \mathrm{Proj}\,\left(P \otimes_{\mathbb{Q}_p} E_h\right) \cong \mathrm{Proj}\,\left(\bigoplus_{n\geq 0} B^{\varphi^h = p^{nh}}\right) \cong \mathrm{Proj}\,\left(\bigoplus_{n\geq 0} B^{\varphi^h = p^n}\right)$$
as desired. □

We invoke the following generalization of Corollary 3.1.10 without proof.

Proposition 3.4.4. Let h and n be positive integers. Every nonzero element $f \in B^{\varphi^h = p^n}$ admits a factorization
$$f = f_1 \cdots f_n \qquad \text{with } f_i \in B^{\varphi^h = p}$$
where the factors are uniquely determined up to $E_h^\times$-multiple.

Remark. Let us briefly sketch the proof of Proposition 3.4.4. The theory of Lubin-Tate formal groups yields a unique 1-dimensional p-divisible formal group law μ_{LT} over $\mathcal{O}_{E_h}$ with $[p]_{\mu_{\mathrm{LT}}}(t) = pt + t^{p^h}$. Denote by G_{LT} the associated p-divisible group over $\mathcal{O}_{E_h}$. By means of the logarithm for G_{LT}, we can construct a group homomorphism
$$\log_h : G_{\mathrm{LT}}(\mathcal{O}_F) := \varprojlim_i G_{\mathrm{LT}}(\mathcal{O}_F/\mathfrak{m}_F^i\mathcal{O}_F) \longrightarrow B^{\varphi^h = p}.$$
It is then not hard to extend the results from §2.3, §2.4, and §3.1 with $\log_h$, $G_{\mathrm{LT}}(\mathcal{O}_F)$, φ^h, ϕ^h, P_h, and X_h respectively taking the roles of log, $1 + \mathfrak{m}_F^*$, φ, ϕ, P, and X. We refer the readers to [**Lur**, Lecture 22-26] for details.

Definition 3.4.5. Let d and h be integers with $h > 0$. We define the *d-th twist* of $\mathcal{O}_{X_h}$ to be the quasicoherent $\mathcal{O}_{X_h}$-module $\mathcal{O}_h(d)$ associated to $P_h(d) := \bigoplus_{n \in \mathbb{Z}} B^{\varphi^h = p^{d+n}}$, where we identify $X_h \cong \text{Proj}\,(P_h)$ as in Proposition 3.4.3.

LEMMA 3.4.6. Let h be a positive integer. For every $d \in \mathbb{Z}$, the $\mathcal{O}_{X_h}$-module $\mathcal{O}_h(d)$ is a line bundle on X_h with a canonical isomorphism $\mathcal{O}_h(d) \cong \mathcal{O}_h(1)^{\otimes d}$.

PROOF. The assertion follows from Proposition 3.4.4 by a general fact as stated in [**Sta**, Tag 01MT]. □

Definition 3.4.7. Let h be a positive integer.

(1) For every positive integer r, we define the *degree r unramified cover* of X_h to be the natural map

$$\pi_{rh,h} : X_{rh} \cong X_h \times_{\text{Spec}\,(E_h)} \text{Spec}\,(E_{rh}) \longrightarrow X_h.$$

(2) For every pair of integers (d, r) with $r > 0$, we write $\mathcal{O}_h(d, r) := (\pi_{rh,h})_* \mathcal{O}_{rh}(d)$.

(3) For every nonzero homogeneous $f \in P$, we denote by $D_h(f)$ the preimage of the open subscheme $D(f) := \text{Spec}\,(B[1/f]^{\varphi=1}) \subseteq X$ under π_h.

LEMMA 3.4.8. Let h be a positive integer.

(1) The scheme X_h is covered by open subschemes of the form $D_h(f)$ for some nonzero homogeneous element $f \in P$.

(2) Given two nonzero homogeneous f and g in P, we have $D_h(f) \cap D_h(g) = D_h(fg)$.

PROOF. Both statements evidently hold for $h = 1$ as we have $X_1 = X = \text{Proj}\,(P)$. The assertion for the general case then follows by the surjectivity of π_h. □

PROPOSITION 3.4.9. Let d, h, and r be integers with $h, r > 0$.

(1) The $\mathcal{O}_{X_h}$-module $\mathcal{O}_h(d, r)$ is a vector bundle on X_h of rank r.

(2) Given a nonzero homogeneous $f \in P$, there exists a canonical identification

$$\mathcal{O}_h(d, r)\,(D_h(f)) \cong B[1/f]^{\varphi^{hr} = p^d}.$$

PROOF. The first statement follows from Lemma 3.4.6 since the morphism $\pi_{rh,h}$ is finite of degree r. The second statement is obvious by construction. □

PROPOSITION 3.4.10. Let d and r be integers with $r > 0$. Given arbitrary positive integers h and n, there exists a natural identification

$$(\pi_{hn,h})^* \mathcal{O}_h(d, r) \cong \mathcal{O}_{hn}(dn, r).$$

PROOF. Let $f \in P$ be an arbitrary nonzero homogeneous element. Since $D_{hn}(f)$ is the inverse image of $D_h(f)$ under $\pi_{hn,h}$, we use Lemma 3.4.2 and Proposition 3.4.9 to find

$$\begin{aligned}
(\pi_{hn,h})^* \mathcal{O}_h(d, r)\,(D_{hn}(f)) &\cong \mathcal{O}_h(d, r)\,(D_h(f)) \otimes_{B[1/f]^{\varphi^h=1}} B[1/f]^{\varphi^{hn}=1} \\
&\cong B[1/f]^{\varphi^{hr}=p^d} \otimes_{B[1/f]^{\varphi^h=1}} \left(B[1/f]^{\varphi^h=1} \otimes_{\mathbb{Q}_p} E_n \right) \\
&\cong B[1/f]^{\varphi^{hr}=p^d} \otimes_{\mathbb{Q}_p} E_n \\
&\cong B[1/f]^{\varphi^{hnr}=p^{dn}} \\
&\cong \mathcal{O}_{hn}(dn, r)\,(D_{hn}(f))\,.
\end{aligned}$$

The desired assertion now follows by Lemma 3.4.8. □

Proposition 3.4.11. Let d and r be integers with $r > 0$. Given arbitrary positive integers h and n, we have a natural isomorphism

$$\mathcal{O}_h(dn, rn) \cong \mathcal{O}_h(d, r)^{\oplus n}.$$

Proof. By Proposition 3.4.10 we obtain a natural isomorphism

$$\mathcal{O}_h(dn, rn) = (\pi_{hr,h})_*(\pi_{hnr,hr})_*\mathcal{O}_{hnr}(dn) \cong (\pi_{hr,h})_*(\pi_{hnr,hr})_*(\pi_{hnr,hr})^*\mathcal{O}_{hr}(d).$$

Then we use the projection formula to find

$$(\pi_{hnr,hr})_*(\pi_{hnr,hr})^*\mathcal{O}_{hr}(d) \cong (\pi_{hnr,hr})_*\mathcal{O}_{X_{hnr}} \otimes_{\mathcal{O}_{X_{hr}}} \mathcal{O}_{hr}(d) \cong \mathcal{O}_{X_{hr}}^{\oplus n} \otimes_{\mathcal{O}_{X_{hr}}} \mathcal{O}_{hr}(d) \cong \mathcal{O}_{hr}(d)^{\oplus n},$$

and consequently deduce the desired assertion. □

Proposition 3.4.12. Let h be a positive integer. We have a canonical isomorphism

$$\mathcal{O}_h(d_1, r_1) \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(d_2, r_2) \cong \mathcal{O}_h(d_1r_2 + d_2r_1, r_1r_2)$$

for all integers d_1, d_2, r_1, r_2 with $r_1, r_2 > 0$.

Proof. Let g and l respectively denote the greatest common divisor and the least common multiple of r_1 and r_2. Since r_1/g and r_2/g are relatively prime integers, the fields E_{r_1h} and E_{r_2h} are linearly disjoint finite extensions of E_{gh} with $E_{r_1h}E_{r_2h} = E_{lh}$. Hence we have an identification $E_{lh} \cong E_{r_1h} \otimes_{E_{gh}} E_{r_2h}$, which gives rise to a cartesian diagram

$$\begin{array}{ccc} X_{lh} & \xrightarrow{\pi_{lh,r_2h}} & X_{r_2h} \\ {\scriptstyle \pi_{lh,r_1h}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{r_2h,gh}} \\ X_{r_1h} & \xrightarrow{\pi_{r_1h,gh}} & X_{gh} \end{array}$$

where all arrows are finite étale. Let us now write $r_1' := r_1/g$ and $r_2' := r_2/g$. Then we find

$$\begin{aligned} \mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_{gh}}} \mathcal{O}_{gh}(d_2, r_2') &= (\pi_{r_1h,gh})_*(\mathcal{O}_{r_1h}(d_1)) \otimes_{\mathcal{O}_{X_{gh}}} (\pi_{r_2h,gh})_*(\mathcal{O}_{r_2h}(d_2)) \\ &\cong (\pi_{lh,gh})_* \Big((\pi_{lh,r_1h})^*\mathcal{O}_{r_1h}(d_1) \otimes_{\mathcal{O}_{X_{lh}}} (\pi_{lh,r_2h})^*\mathcal{O}_{r_2h}(d_2)\Big) \\ &\cong (\pi_{lh,gh})_* \Big(\mathcal{O}_{lh}(d_1r_1') \otimes_{\mathcal{O}_{X_{lh}}} \mathcal{O}_{lh}(d_2r_2')\Big) \\ &\cong (\pi_{lh,gh})_*\mathcal{O}_{lh}(d_1r_1' + d_2r_2') \\ &= \mathcal{O}_{gh}(d_1r_1' + d_2r_2', r_1'r_2') \end{aligned}$$

where the isomorphisms respectively follow from the Künneth formula, Proposition 3.4.10, and Lemma 3.4.6. We thus use the projection formula, Proposition 3.4.10, and Proposition 3.4.11 to obtain an identification

$$\begin{aligned} \mathcal{O}_h(d_1, r_1) \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(d_2, r_2) &= (\pi_{gh,h})_*\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(d_2, r_2) \\ &\cong (\pi_{gh,h})_* \Big(\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_{gh}}} (\pi_{gh,h})^*\mathcal{O}_h(d_2, r_2)\Big) \\ &\cong (\pi_{gh,h})_* \Big(\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_{gh}}} \mathcal{O}_{gh}(d_2g, r_2)\Big) \\ &\cong (\pi_{gh,h})_* \Big(\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_{gh}}} \mathcal{O}_{gh}(d_2, r_2')^{\oplus g}\Big) \\ &\cong (\pi_{gh,h})_*\mathcal{O}_{gh}(d_1r_1' + d_2r_2', r_1'r_2')^{\oplus g} \\ &= \mathcal{O}_h(d_1r_1' + d_2r_2', gr_1r_2)^{\oplus g} \\ &\cong \mathcal{O}_h(d_1r_1 + d_2r_2, r_1r_2), \end{aligned}$$

thereby completing the proof. □

PROPOSITION 3.4.13. Let d and r be ingeters with $r > 0$. For every positive integer h, there exists a canonical isomorphism

$$\mathcal{O}_h(d,r)^\vee \cong \mathcal{O}_h(-d,r).$$

PROOF. Proposition 3.4.11 and Proposition 3.4.12 together yield a natural map

$$\mathcal{O}_h(d,r) \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(-d,r) \cong \mathcal{O}_{X_h}^{\oplus r^2} \cong \mathcal{O}_{X_h}^{\oplus r} \otimes_{\mathcal{O}_{X_h}} \left(\mathcal{O}_{X_h}^{\oplus r}\right)^\vee \longrightarrow \mathcal{O}_{X_h}$$

where the last arrow is given by the trace map. It is straightforward to verify that this map is a perfect pairing, which in turn yields the desired isomorphism. □

PROPOSITION 3.4.14. Let d and r be integers with $r > 0$.

(1) The vector bundle $\mathcal{O}(d,r) := \mathcal{O}_1(d,r)$ on X is semistable of rank r and degree d.
(2) If d and r are relatively prime, then the bundle $\mathcal{O}(d,r)$ is stable.

PROOF. Proposition 3.4.11 and Proposition 3.4.12 together yield a natural isomorphism

$$\mathcal{O}(d,r)^{\otimes r} \cong \mathcal{O}(dr^r, r^r) \cong \mathcal{O}(d)^{\oplus r^r}. \tag{3.18}$$

Moreover, we find $\deg\left(\mathcal{O}(d)^{\oplus r^r}\right) = dr^r$ by Proposition 3.3.4. Therefore it follows by Proposition 3.3.6 and Proposition 3.4.9 that $\mathcal{O}(d,r)$ is of rank r and degree d. Furthermore, since $\mathcal{O}(d)$ is stable as noted in Example 3.3.14, we find by Proposition 3.3.18 that $\mathcal{O}(d)^{\oplus r^r}$ is semistable, and consequently deduce by (3.18) and Proposition 3.3.17 that $\mathcal{O}(d,r)$ is semistable as well.

Let us now assume that d and r are relatively prime. Take an arbitrary nonzero proper subbundle $\mathcal{V}$ of $\mathcal{O}(d,r)$. We have $\mu(\mathcal{V}) \neq d/r$ as $\mathrm{rk}(\mathcal{V})$ is less than $\mathrm{rk}(\mathcal{O}(d,r)) = r$. Hence we find $\mu(\mathcal{V}) < \lambda$ by the semistability of $\mathcal{O}(d,r)$, thereby deducing that $\mathcal{O}(d,r)$ is stable. □

Remark. Proposition 3.4.14 readily extends to $\mathcal{O}_h(d,r)$ and X_h for every positive integer h, as it turns out that X_h is a complete algebraic curve. In fact, extending the remark after Proposition 3.4.4, it is not hard to show that all results from §3.2 remain valid with φ^h, P_h, X_h, and $\mathcal{O}_h(d)$ respectively in place of φ, P, X, and $\mathcal{O}(d)$; in particular, X_h is a Dedekind scheme whose Picard group is isomorphic to $\mathbb{Z}$.

Definition 3.4.15. Let $\lambda = d/r$ be a rational number, written in a reduced form with $r > 0$. We refer to $\mathcal{O}(\lambda) := \mathcal{O}_1(d,r)$ as the *canonical stable bundle* on X of slope λ.

PROPOSITION 3.4.16. Let λ be a rational number.

(1) There exists a canonical isomorphism $\mathcal{O}(\lambda)^\vee \cong \mathcal{O}(-\lambda)$.
(2) Given a rational number λ', we have a natural isomorphism

$$\mathcal{O}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{O}(\lambda') \cong \mathcal{O}(\lambda + \lambda')^{\oplus n}$$

for some positive integer n.

PROOF. The first statement is a special case of Proposition 3.4.13. The second statement follows from Proposition 3.4.11 and Proposition 3.4.12. □

Remark. By the remark after Proposition 3.4.14, for every positive integer h we can define the canonical stable bundle $\mathcal{O}_h(\lambda)$ of slope λ on X_h and extend Proposition 3.4.16 to $\mathcal{O}_h(\lambda)$.

3.5. Classification of the vector bundles

In this subsection, we describe a complete classification of vector bundles on the Fargues-Fontaine curve. We invoke the following technical result without proof.

PROPOSITION 3.5.1. Let λ be a rational number.

(1) A vector bundle on X is semistable of slope λ if and only if is isomorphic to $\mathcal{O}(\lambda)^{\oplus n}$ for some $n \geq 1$.
(2) If we have $\lambda \geq 0$, the cohomology group $H^1(X, \mathcal{O}(\lambda))$ vanishes.

Remark. The second statement is relatively easy to prove. Let us write $\lambda = d/r$ where d and r are relatively prime integers with $r > 0$. As remarked after Proposition 3.4.14, Theorem 3.2.9 is valid with $\mathcal{O}_r(d)$ and X_r respectively in place of $\mathcal{O}(d)$ and X. Hence for $\lambda \geq 0$ we find

$$H^1(X, \mathcal{O}(\lambda)) = H^1(X, (\pi_r)_*\mathcal{O}_r(d)) \cong H^1(X_r, \mathcal{O}_r(d)) = 0.$$

On the other hand, the first statement is one of the most technical results from the original work of Fargues and Fontaine [**FF18**]. Here we can only sketch some key ideas for the proof. We refer the curious readers to [**FF14**, §6] for a good exposition of the proof.

The if part of the first statement is immediate by Proposition 3.4.14. In order to prove the converse, it is essential to simultaneously consider all unramified covers of X; more precisely, we assert that every semistable vector bundle $\mathcal{V}$ on X_h of slope λ is isomorphic to $\mathcal{O}_h(\lambda)^{\oplus n}$ for some $n \geq 1$, where we set $\mathcal{O}_h(\lambda) := \mathcal{O}_h(d, r)$. The proof of this statement is given by a series of dévissage arguments as follows:

(a) We may replace $\mathcal{V}$ with $(\pi_{rh,r})^*\mathcal{V}$ to assume that λ is an integer; this reduction is based on the identification $(\pi_{rh,r})_*(\pi_{rh,r})^*\mathcal{O}_h(\lambda) \cong \mathcal{O}_h(d)^{\oplus r}$ given by Proposition 3.4.10 and the fact that $(\pi_{rh,r})^*\mathcal{V}$ is semistable of slope d as seen by an elementary Galois descent argument based on Theorem 3.3.22.
(b) We may replace $\mathcal{V}$ by $\mathcal{V}(-\lambda) := \mathcal{V} \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(-\lambda)$ to further assume $\lambda = 0$; this reduction is based on the identification $\mathcal{O}_h(\lambda) \cong \mathcal{O}_h \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h$ and the fact that $\mathcal{V}(-\lambda)$ is semistable of slope 0 as easily seen by Proposition 3.3.6.
(c) With $\lambda = 0$, it suffices to prove that $H^0(X_h, \mathcal{V})$ does not vanish; indeed, any nonzero global section of $\mathcal{V}$ gives rise to an exact sequence of vector bundles on X_h
$$0 \longrightarrow \mathcal{O}_{X_h} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0$$
where $\mathcal{W}$ is semistable of slope 0 by Proposition 3.3.18, thereby allowing us to proceed by induction on $\mathrm{rk}(\mathcal{V})$ with the identification $\mathrm{Ext}^1_{\mathcal{O}_{X_h}}(\mathcal{O}_h, \mathcal{O}_h) \cong H^1(X_h, \mathcal{O}_{X_h}) = 0$.
(d) The proof further reduces to the case where $\mathcal{V}$ fits into a short exact sequence
$$0 \longrightarrow \mathcal{O}_h(-1/n) \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}_h(1) \longrightarrow 0$$
with $n = \mathrm{rk}(\mathcal{V}) - 1$; this reduction involves a generalization of Grothendieck's argument for the classification of vector bundles on the projective line.
(e) The exact sequence above turns out to naturally arise from p-divisible groups, as we will remark after Example 3.5.4; as a consequence the assertion eventually follows from some results about period morphisms on the Lubin-Tate spaces due to Drinfeld [**Dri76**], Gross-Hopkins [**GH94**], and Laffaille [**Laf85**].

COROLLARY 3.5.2. The tensor product of two semistable vector bundles on X is semistable.

PROOF. This is an immediate consequence of Proposition 3.4.16 and Proposition 3.5.1. □

THEOREM 3.5.3 (Fargues-Fontaine [**FF18**]). Every vector bundle $\mathcal{V}$ on X admits a unique Harder-Narasimhan filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V},$$

which (noncanonically) splits into a direct sum decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{n} \mathcal{O}(\lambda_i)^{\oplus m_i}$$

where we set $\lambda_i := \mu(\mathcal{V}_i/\mathcal{V}_{i-1})$ for each $i = 1, \cdots, n$.

PROOF. Existence and uniqueness of the Harder-Narasimhan filtration is an immediate consequence of Theorem 3.3.22. Hence it remains to prove that the Harder-Narasimhan filtration splits. Let us proceed by induction on n. If we have $n = 0$, then the assertion is trivial. We henceforth assume $n > 0$. By construction each successive quotient $\mathcal{V}_i/\mathcal{V}_{i-1}$ is semistable of slope λ_i. Hence Proposition 3.5.1 yields an isomorphism

$$\mathcal{V}_i/\mathcal{V}_{i-1} \simeq \mathcal{O}(\lambda_i)^{\oplus m_i} \qquad \text{for each } i = 1, \cdots, n \tag{3.19}$$

where m_i is a positive integer. Moreover, by the induction hypothesis, the filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_{n-1}$$

splits into a direct sum decomposition

$$\mathcal{V}_{l-1} \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}(\lambda_i)^{\oplus m_i}. \tag{3.20}$$

Hence it suffices to establish the identity

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{V}/\mathcal{V}_{n-1}, \mathcal{V}_{n-1}) = 0. \tag{3.21}$$

For each $i = 1, \cdots, n$, Proposition 3.4.16 yields an identification

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}(\lambda_n), \mathcal{O}(\lambda_i)) \cong H^1(X, \mathcal{O}(\lambda_i) \otimes_{\mathcal{O}_X} \mathcal{O}(\lambda_n)^\vee) \cong H^1(X, \mathcal{O}(\lambda_i - \lambda_n)^{\oplus n_i})$$

where n_i is a positive integer. Since we have $\lambda_i \geq \lambda_n$ for each $i = 1, \cdots, n$, we find

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}(\lambda_n), \mathcal{O}(\lambda_i)) = 0 \qquad \text{for each } i = 1, \cdots, n$$

by Proposition 3.5.1. Therefore we deduce the identity (3.21) by the decompositions (3.19) and (3.20), thereby completing the proof. □

Remark. Theorem 3.5.3 is an analogue of the fact that every vector bundle $\mathcal{W}$ on the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ admits a direct sum decomposition

$$\mathcal{W} \simeq \bigoplus_{j=1}^{l} \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d_j)^{\oplus k_j} \qquad \text{with } d_j \in \mathbb{Z}.$$

The only essential difference is that semistable vector bundles on X may have rational slopes, whereas semistable vector bundles on $\mathbb{P}^1_{\mathbb{C}}$ have integer slopes. This difference comes from the fact that we have $H^1(X, \mathcal{O}(-1)) \neq 0$ and $H^1(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)) = 0$ as remarked after Theorem 3.2.9.

It is worthwhile to mention that an equivalent result of Theorem 3.5.3 was first obtained by Kedlaya [**Ked05**]. In fact, Kedlaya's result can be reformulated as a classification of vector bundles on the adic Fargues-Fontaine curve, which recovers Theorem 3.5.3 by Theorem 1.3.24.

Example 3.5.4. Let us write $W(\overline{\mathbb{F}}_p)$ for the ring of Witt vectors over $\overline{\mathbb{F}}_p$, and K_0 for the fraction field of $W(\overline{\mathbb{F}}_p)$. Let N be an isocrystal over K_0 which admits a decomposition

$$N \simeq \bigoplus_{i=1}^{n} N(\lambda_i)^{\oplus m_i} \qquad \text{with } \lambda_i \in \mathbb{Q}. \tag{3.22}$$

We assert that N naturally gives rise to a vector bundle $\mathcal{E}(N)$ on X with an isomorphism

$$\mathcal{E}(N) \simeq \bigoplus_{i=1}^{n} \mathcal{O}(\lambda_i)^{\oplus m_i}. \tag{3.23}$$

We may regard K_0 as a subring of B under the identification

$$K_0 = W(\overline{\mathbb{F}}_p)[1/p] \cong \left\{ \sum [c_n]p^n \in A_{\text{inf}}[1/p] : c_n \in \overline{\mathbb{F}}_p \right\}.$$

Then by construction φ restricts to the Frobenius automorphism of K_0, and thus acts on N and $N^\vee$ via the Frobenius automorphisms φ_N and $\varphi_{N^\vee}$. Hence we get a graded P-module

$$P(N) := \bigoplus_{n \geq 0} (N^\vee \otimes_{K_0} B)^{\varphi = p^n}.$$

Let us set $\mathcal{E}(N)$ to be the associated quasicoherent sheaf on X, and take an arbitrary nonzero homogeneous element $f \in P$. In addition, for each $i = 1, \cdots, n$, we write $\lambda_i := d_i/r_i$ where d_i and r_i are relatively prime integers with $r_i > 0$. By construction we have

$$\begin{aligned} \mathcal{E}(N)(D(f)) &\cong \left(N^\vee \otimes_{K_0} B[1/f]\right)^{\varphi=1} = (\text{Hom}_{K_0}(N, K_0) \otimes_{K_0} B[1/f])^{\varphi=1} \\ &\cong \text{Hom}_{K_0}(N, B[1/f])^{\varphi=1}. \end{aligned} \tag{3.24}$$

Moreover, since each $N(\lambda_i)$ admits a basis $(\varphi^j(n))$ for some $n \in N(\lambda_i)$ with $\varphi^{r_i}(n) = p^{d_i}n$, there exists an identification

$$\text{Hom}_{K_0}(N(\lambda_i), B[1/f])^{\varphi=1} \cong B[1/f]^{\varphi^{r_i} = p^{d_i}} \cong \mathcal{O}(\lambda_i)(D(f)) \tag{3.25}$$

where the last isomorphism follows from Proposition 3.4.9. As $f \in P$ is arbitrarily chosen, we obtain the isomorphism (3.23) by (3.22), (3.24) and (3.25).

Remark. As noted in Chapter II, Theorem 2.3.24, every isocrystal over K_0 admits a direct sum decomposition as in (3.22). Hence by Theorem 3.5.3 and Example 3.5.4 we obtain an essentially surjective functor

$$\mathcal{E} : \varphi\text{−}\text{Mod}_{K_0} \longrightarrow \text{Bun}_X$$

where $\varphi\text{−}\text{Mod}_{K_0}$ and Bun_X respectively denote the category of isocrystals over K_0 and the category of vector bundles on X. Furthremore, if we have $0 \leq \lambda_i \leq 1$ for each $i = 1, \cdots, n$, it turns out that there exists a p-divisible group G over $\overline{\mathbb{F}}_p$ with

$$\mathcal{E}(\mathbb{D}(G)[1/p]) \simeq \bigoplus_{i=1}^{n} \mathcal{O}(\lambda_i)^{\oplus m_i}.$$

However, the functor $\mathcal{E}$ is not an equivalence of categories; indeed, for arbitrary rational numbers κ and λ with $\kappa < \lambda$, we have

$$\text{Hom}_{\varphi-\text{Mod}_{K_0}}(N(\kappa), N(\lambda)) = 0 \quad \text{and} \quad \text{Hom}_{\mathcal{O}_X}(\mathcal{E}(N(\kappa)), \mathcal{E}(N(\lambda))) \neq 0.$$

4. Applications to p-adic representations

In this section, we prove some fundamental results about p-adic representations and period rings by exploiting our accumulated knowledge of the Fargues-Fontaine curve. The primary references for this section are Fargues and Fontaine's survey paper [**FF12**] and Morrow's notes [**Mor**].

4.1. Geometrization of p-adic period rings

Throughout this section, we let K be a p-adic field with the absolute Galois group Γ_K, the inertia group I_K and the residue field k. We also write $W(k)$ for the ring of Witt vectors over k, and K_0 for its fraction field.

PROPOSITION 4.1.1. The tilt of $\mathbb{C}_K$ is algebraically closed.

PROOF. Let $f(x)$ be an arbitrary monic polynomial of degree $d > 0$ over $\mathbb{C}_K^\flat$. We wish to show that $f(x)$ has a root in $\mathbb{C}_K^\flat$. Take an element m in the maximal ideal of $\mathcal{O}_{\mathbb{C}_K^\flat}$. We may replace $f(x)$ by $m^{nd}f(x/m^n)$ for some sufficiently large n to assume that $f(x)$ is a polynomial over $\mathcal{O}_{\mathbb{C}_K^\flat}$. Moreover, we may assume $d > 1$ since otherwise the assertion would be obvious. Let us now write

$$f(x) = x^d + c_1x^{d-1} + \cdots + c_d \qquad \text{with } c_i \in \mathcal{O}_{\mathbb{C}_K^\flat}.$$

Proposition 2.1.6 and Proposition 2.1.7 from Chapter III together yield an identification

$$\mathcal{O}_{\mathbb{C}_K^\flat} \cong \varprojlim_{c \mapsto c^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}. \tag{4.1}$$

Write $(c_{i,n})$ for the image of each c_i under this isomorphism, and choose a lift $\widetilde{c_{i,n}} \in \mathcal{O}_{\mathbb{C}_K}$ of each $c_{i,n}$. In addition, for each $n \geq 0$ we set

$$f_n(x) := x^d + c_{1,n}x^{d-1} + \cdots + c_{d,n} \qquad \text{and} \qquad \widetilde{f_n}(x) := x^d + \widetilde{c_{1,n}}x^{d-1} + \cdots + \widetilde{c_{d,n}}.$$

Then for each $n \geq 1$ we have

$$f_{n-1}(x^p) = x^{dp} + c_{1,n}^p x^{(d-1)p} + \cdots + c_{d,n}^p = \left(x^d + c_{1,n}x^{d-1} + \cdots + c_{d,n}\right)^p = f_n(x)^p. \tag{4.2}$$

Moreover, since $\mathbb{C}_K$ is algebraically closed as noted in Chapter II, Proposition 3.1.13, each $\widetilde{f_n}(x)$ admits a factorization

$$\widetilde{f_n}(x) = (x - \alpha_{n,1}) \cdots (x - \alpha_{n,d}) \qquad \text{with } \alpha_{n,j} \in \mathcal{O}_{\mathbb{C}_K}.$$

Let us denote by $\overline{\alpha_{n,j}}$ the image of each $\alpha_{n,j}$ under the natural surjection $\mathcal{O}_{\mathbb{C}_K} \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. For each $\overline{\alpha_{n,j}}$ with $n \geq 1$ we obtain $f_{n-1}(\overline{\alpha_{n,j}}^p) = f_n(\overline{\alpha_{n,j}})^p = 0$ by (4.2), and in turn find

$$\widetilde{f_{n-1}}(\alpha_{n,j}^p) = (\alpha_{n,j}^p - \alpha_{n-1,1}) \cdots (\alpha_{n,j}^p - \alpha_{n-1,d}) \in p\mathcal{O}_{\mathbb{C}_K}.$$

Hence for each $\alpha_{n,j}$ with $n \geq 1$ we have $\alpha_{n,j}^p - \alpha_{n-1,l} \in p^{1/d}\mathcal{O}_{\mathbb{C}_K}$ for some l, and consequently obtain $\overline{\alpha_{n,j}}^{p^d} = \overline{\alpha_{n-1,l}}^{p^{d-1}}$ by Proposition 2.1.6 in Chapter III. It follows that there exists a sequence of integers (j_n) with $\overline{\alpha_{n,j_n}}^{p^d} = \overline{\alpha_{n-1,j_{n-1}}}^{p^{d-1}}$ for all $n \geq 1$. Let us now set $\overline{\alpha} := \left(\overline{\alpha_{n+d-1,j_{n+d-1}}}^{p^{d-1}}\right)$. Then under the identification (4.1) we find

$$f(\overline{\alpha}) = \left(f_n\left(\overline{\alpha_{n+d-1,j_{n+d-1}}}^{p^{d-1}}\right)\right) = \left(f_{n+d-1}\left(\overline{\alpha_{n+d-1,j_{n+d-1}}}\right)\right) = 0$$

where the second identity follows by (4.2). □

Remark. Our proof above readily extends to show that the tilt of an algebraically closed perfectoid field is algebraically closed.

For the rest of this section, we take $F = \mathbb{C}_K^\flat$ and regard $\mathbb{C}_K$ as an untilt of F. We also fix an element $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\sharp = p$ and set $\xi := [p^\flat] - p \in A_{\text{inf}}$. In addition, we choose a valuation ν_F on F with $\nu_F(p^\flat) = 1$.

Proposition 4.1.2. Let ε be an element in $\mathcal{O}_F$ with $\varepsilon^\sharp = 1$ and $(\varepsilon^{1/p})^\sharp \neq 1$.

(1) We have $\varepsilon \in 1 + \mathfrak{m}_F^*$.
(2) The element $t := \log(\varepsilon) \in B^{\varphi=p}$ is a prime in P, and gives rise to a closed point ∞ on X with the following properties:
  (i) The residue field at ∞ is naturally isomorphic to $\mathbb{C}_K$.
  (ii) The completed local ring at ∞ is naturally isomorphic to B_{dR}^+.

Proof. The first statement is an immediate consequence of Lemma 2.2.21 from Chapter III (or the proof of Proposition 2.3.3). We then observe by Proposition 2.3.3 that $t = \log(\varepsilon)$ vanishes at an element $y_\infty \in Y$ represented by $\mathbb{C}_K$, and consequently deduce the second statement from Proposition 2.4.7 and Theorem 2.4.8. □

Proposition 4.1.3. There exists a natural isomorphism

$$B_{\text{dR}}^+ \cong \varprojlim_j B/\ker(\widehat{\theta_{\mathbb{C}_K}})^j \tag{4.3}$$

which induces a topology on B_{dR}^+ with the following properties:

(i) The subring A_{inf} of B_{dR}^+ is closed.
(ii) The map $\theta_{\mathbb{C}_K}[1/p] : A_{\text{inf}}[1/p] \twoheadrightarrow \mathbb{C}_K$ induced by $\theta_{\mathbb{C}_K}$ is continuous and open with respect to the p-adic topology on $\mathbb{C}_K$.
(iii) The logarithm on $1 + \mathfrak{m}_F$ induces a continuous map $\log : \mathbb{Z}_p(1) \longrightarrow B_{\text{dR}}^+$ under the natural identification $\mathbb{Z}_p(1) = \varprojlim \mu_{p^v}(\overline{K}) = \{ \varepsilon \in \mathcal{O}_F : \varepsilon^\sharp = 1 \}$.
(iv) The multiplication by any uniformizer yields a closed embedding on B_{dR}^+.
(v) The ring B_{dR}^+ is complete.

Proof. The natural isomorphism (4.3) is given by Proposition 2.2.7. Let us equip B_{dR}^+ with the inverse limit topology via (4.3). The property (ii) follows from Proposition 1.2.16 and the fact that $\theta_{\mathbb{C}_K}[1/p]$ extends to $\widehat{\theta_{\mathbb{C}_K}}$. The property (iii) is evident by Proposition 3.1.8.

Let us now establish the property (i). Recall that we may regard $A_{\text{inf}}[1/p]$ as a subring of B_{dR}^+ in light of Proposition 2.2.18 from Chapter III. Proposition 3.1.4 implies that A_{inf} is complete with respect to all Gauss norms. Moreover, by Example 2.1.6 we have $|\xi|_\rho < 1$ for all $\rho \in (0, 1)$, and consequently find that every ξ-adically Cauchy sequence in A_{inf} is also Cauchy with respect to all Gauss norms. We then deduce the assertion by the fact that ξ generates $\ker(\widehat{\theta_{\mathbb{C}_K}})$ as noted in Corollary 2.2.4.

It remains to verify the properties (iv) and (iv). We find by Proposition 1.2.16 that $\ker(\widehat{\theta_{\mathbb{C}_K}}) = \xi B$ is closed in B, and in turn deduce that $\ker(\widehat{\theta_{\mathbb{C}_K}})^j = \xi^j B$ is closed in B for each $j \geq 1$. Hence the property (iv) follows by the fact that every uniformizer of B_{dR}^+ is a unit multiple of ξ as noted in Proposition 2.2.7. In addition, we find by the completeness of B that $B/\ker(\widehat{\theta_{\mathbb{C}_K}})^j$ is complete for each $j \geq 1$, and consequently obtain the property (iv). □

Remark. Proposition 4.1.3 proves Proposition 2.2.19 from Chapter III. Our proof does not rely on any unproved results such as Proposition 2.4.1, Proposition 3.4.4 or Proposition 3.5.1.

We henceforth fix $\varepsilon \in 1+\mathfrak{m}_F^*$, $t \in B^{\varphi=p}$ and $\infty \in |X|$ as in Proposition 4.1.2. We also write B^+ for the closure of $A_{\text{inf}}[1/p]$ in B. In addition, for every $\rho \in (0,1)$ we denote by B_ρ^+ the closure of $A_{\text{inf}}[1/p]$ in $B_{[\rho,\rho]}$.

LEMMA 4.1.4. Let V be a normed space over $\mathbb{Q}_p$, and let $\widehat{V_0}$ denote the p-adic completion of the closed unit disk V_0 in V. The completion of V with respect to its norm is naturally isomorphic to $\widehat{V_0}[1/p]$.

PROOF. Since p is topologically nilpotent in $\mathbb{Q}_p$, we have a neighborhood basis for $0 \in V$ given by the sets $p^n V_0$ for $n \geq 0$. This implies that a sequence in V_0 is Cauchy with respect to the norm on V if and only if it is p-adically Cauchy. Hence $\widehat{V_0}$ coincides with the completion of V_0 with respect to the norm on V. The assertion now follows by the fact that every Cauchy sequence in V becomes a Cauchy sequence in V_0 after a multiplication by some power of p. □

Remark. The notion of p-adic completion is not meaningful for V, as we have $p^n V = V$ for all $n \geq 0$.

PROPOSITION 4.1.5. Let c be an element in $\mathcal{O}_F^\times$. There exists a canonical continuous isomorphism

$$B_{|c|}^+ \cong \widehat{A_{\text{inf}}[[c]/p]}[1/p]$$

where $\widehat{A_{\text{inf}}[[c]/p]}$ denotes the p-adic completion of $A_{\text{inf}}[[c]/p]$.

PROOF. By construction, the topological ring $B_{|c|}^+$ is naturally isomorphic to the completion of $A_{\text{inf}}[1/p]$ with respect to the Gauss $|c|$-norm. In light of Lemma 4.1.4, it is thus sufficient to establish the identification

$$A_{\text{inf}}[[c]/p] = \left\{ f \in A_{\text{inf}}[1/p] : |f|_{|c|} \leq 1 \right\}.$$

Since we have $|[c]/p|_{|c|} = 1$, the ring $A_{\text{inf}}[[c]/p]$ is contained in the set on the right hand side. Let us now consider an arbitrary element $f \in A_{\text{inf}}[1/p]$ with $|f|_{|c|} \leq 1$. We wish to show that f belongs to $A_{\text{inf}}[[c]/p]$. Let us write the Teichmüller expansion of f as

$$f = \sum_{n<0} [c_n]p^n + \sum_{n \geq 0} [c_n]p^n \qquad \text{with } c_n \in \mathcal{O}_F \tag{4.4}$$

where the first summation on the right hand side contains only finitely many nonzero terms. For every $n \in \mathbb{Z}$ we find $|c_n|\,|c|^n \leq |f|_{|c|} = 1$, or equivalently $|c_n| \leq |c|^{-n}$. Hence for every $n < 0$ we have $c_n = c^{-n} d_n$ for some $d_n \in \mathcal{O}_F$, and consequently obtain

$$[c_n]p^n = [d_n] \cdot ([c]/p)^{-n} \in A_{\text{inf}}[[c]/p].$$

The assertion is now evident by (4.4). □

Remark. Given two elements $c, d \in \mathcal{O}_F^\times$ with $|c| \leq |d|$, we can argue as above to obtain an identification

$$B_{[|c|,|d|]} \cong \widehat{A_{\text{inf}}[[c]/p, p/[d]]}[1/p]$$

where $\widehat{A_{\text{inf}}[[c]/p, p/[d]]}$ denotes the p-adic completion of $A_{\text{inf}}[[c]/p, p/[d]]$. This is in some sense reminiscent of our discussion in Example 1.3.13, which shows that for arbitrary positive real numbers $i, j \in \mathbb{Z}[1/p]$ the ring $B_{[|\varpi|^i, |\varpi|^j]}$ coincides with the completion of $A_{\text{inf}}[1/p, 1/[\varpi]]$ with respect to the ideal I generated by $[\varpi^i]/p$ and $p/[\varpi^j]$. We can use the above identification to show that the natural map $B \longrightarrow B_{\text{dR}}^+$ extends to a map $B_{[a,b]} \longrightarrow B_{\text{dR}}^+$ for any closed interval $[a,b] \subseteq (0,1)$.

PROPOSITION 4.1.6. We have natural continuous embeddings

$$B^+_{1/p^p} \hookrightarrow B^+_{\text{cris}} \hookrightarrow B^+_{1/p}.$$

PROOF. Let A^0_{cris} be the A_{inf}-subalgebra in $A_{\text{inf}}[1/p]$ generated by the elements of the form $\xi^n/n!$ with $n \geq 0$. By definition we have $B^+_{\text{cris}} = A_{\text{cris}}[1/p]$, where A_{cris} is naturally isomorphic to the p-adic completion of A^0_{cris} as noted in Chapter III, Proposition 3.1.9. Moreover, Proposition 4.1.5 yields natural identifications

$$B^+_{1/p^p} \cong \widehat{A_{\text{inf}}[[(p^\flat)^p]/p]}[1/p] \qquad \text{and} \qquad B^+_{1/p} \cong \widehat{A_{\text{inf}}[[p^\flat]/p]}[1/p],$$

where $\widehat{A_{\text{inf}}[[(p^\flat)^p]/p]}$ and $\widehat{A_{\text{inf}}[[p^\flat]/p]}$ respectively denote the p-adic completions of $A_{\text{inf}}[[(p^\flat)^p]/p]$ and $A_{\text{inf}}[[p^\flat]/p]$. Hence it suffices to show

$$A_{\text{inf}}[[(p^\flat)^p]/p] \subseteq A^0_{\text{cris}} \subseteq A_{\text{inf}}[[p^\flat]/p]. \tag{4.5}$$

We obtain the first inclusion in (4.5) by observing

$$\frac{[p^\flat]^p}{p} = \frac{(\xi+p)^p}{p} = (p-1)! \cdot \frac{\xi}{p!} + \sum_{i=1}^{p} \binom{p}{i} p^{i-1}\xi^{p-i} \in A^0_{\text{cris}}.$$

In addition, we find

$$\frac{\xi^n}{n!} = \frac{([p^\flat]-p)^n}{n!} = \frac{p^n}{n!}\left(\frac{[p^\flat]}{p} - 1\right)^n \in A_{\text{inf}}[[p^\flat]/p] \qquad \text{for all } n \geq 0$$

as $p^n/n!$ is an element of $\mathbb{Z}_p$, and consequently deduce the second inclusion in (4.5). □

LEMMA 4.1.7. Let $[a,b]$ be a closed subinterval of $(0,1)$. There exists some $e > 0$ with

$$|f|_b \leq |f|_a^e \qquad \text{for every } f \in A_{\text{inf}}[1/p].$$

PROOF. Let us set $l := -\log_p(b)$ and $r := -\log_p(a)$. Since $\mathcal{L}_f$ is a concave piecewise linear function as noted in Corollary 2.1.11, its graph on $(0,l]$ should be bounded above by the line which passes through the points $(l, \mathcal{L}_f(l))$ and $(r, \mathcal{L}_f(r))$. Hence we have

$$\mathcal{L}_f(s) \leq \frac{\mathcal{L}_f(r)-\mathcal{L}_f(l)}{r-l}(s-l) + \mathcal{L}_l \qquad \text{for all } s \in (0,l],$$

and consequently find

$$\lim_{s\to 0} \mathcal{L}_f(s) \leq \frac{-l(\mathcal{L}_f(r)-\mathcal{L}_f(l))}{r-l} + \mathcal{L}_l = \frac{-l\mathcal{L}_f(r) + r\mathcal{L}_f(l)}{r-l}.$$

Meanwhile, Proposition 3.1.4 yields an integer n with

$$\mathcal{L}_f(s) = -\log_p\left(|f|_{p^{-s}}\right) \geq -\log_p(p^{-ns}) = ns \qquad \text{for all } s \in (0,\infty),$$

and in turn implies $\lim\limits_{s\to 0} \mathcal{L}_f(s) \geq 0$. We thus obtain $r\mathcal{L}_f(l) \geq l\mathcal{L}_f(r)$, and consequently find

$$|f|_b = p^{-\mathcal{L}_f(r)} \leq p^{-(r/l)\mathcal{L}_f(l)} = |f|_a^{r/l}$$

as desired. □

PROPOSITION 4.1.8. For every closed interval $[a,b] \subseteq (0,1)$, there exists a canonical continuous embedding $B^+_a \hookrightarrow B^+_b$.

PROOF. Lemma 4.1.7 implies that every Cauchy sequence in $A_{\text{inf}}[1/p]$ with respect to the Gauss a-norm is Cauchy with respect to the Gauss b-norm. Hence the assertion is evident by construction. □

For the rest of this section, we write $\widetilde{B^+} := \varinjlim B_\rho^+$ where the transition maps are the natural injective maps given by Proposition 4.1.8, and regard each B_ρ^+ as a subring of $\widetilde{B^+}$. We also regard B_{cris}^+ as a subring of $\widetilde{B^+}$ in light of Proposition 4.1.6.

Proposition 4.1.9. The Frobenius automorphism of $A_{\text{inf}}[1/p]$ uniquely extends to an automorphism φ^+ of $\widetilde{B^+}$ with the following properties:

(i) φ and φ^+ agree on B^+.
(ii) The Frobenius endomorphism of B_{cris} and φ^+ agree on B_{cris}^+.
(iii) φ^+ restricts to an isomorphism $B_\rho^+ \simeq B_{\rho^p}^+$ for every $\rho \in (0,1)$.

Proof. Let φ_{inf} denote the Frobenius automorphism of $A_{\text{inf}}[1/p]$. Then we have

$$\varphi_{\text{inf}}\left(\sum [c_n]p^n\right) = \sum [c_n^p]p^n \qquad \text{for all } c_n \in \mathcal{O}_F,$$

and consequently find

$$|\varphi_{\text{inf}}(f)|_{\rho^p} = |f|_\rho^p \qquad \text{for all } f \in A_{\text{inf}}[1/p] \text{ and } \rho \in (0,1).$$

It follows by Lemma 1.2.15 that φ_{inf} uniquely extends to a continuous ring isomorphism $\varphi_\rho^+ : B_\rho^+ \simeq B_{\rho^p}^+$ for each $\rho \in (0,1)$. For every closed subinterval $[a,b]$ of $(0,1)$, the restriction of φ_b^+ on B_a^+ is a continuous extension of φ_{inf}, and thus agrees with φ_a^+. Hence we obtain an isomorphism

$$\varphi^+ : \widetilde{B^+} = \varinjlim B_\rho^+ \simeq \varinjlim B_{\rho^p}^+ = \widetilde{B^+}.$$

It is evident by construction that φ^+ is an extension of φ_{inf} and each B_ρ^+ with $\rho \in (0,1)$. The uniqueness of each φ_ρ^+ implies that φ^+ is a unique extension of φ_{inf} with the property (iii). Moreover, the restriction of φ^+ on B_{cris}^+ is a continuous extension of φ_{inf}, and thus agrees with the Frobenius endomorphism on B_{cris}^+ by Lemma 3.1.10 from Chapter III.

It remains to verify the property (i) of φ^+. By construction, both φ and φ^+ extend φ_{inf}. In addition, the property (iii) implies that φ^+ restricts to an isomorphism

$$B^+ = \varprojlim B_\rho^+ \simeq \varprojlim B_{\rho^p}^+ = B^+$$

where the transition maps in each limit are the natural inclusions. Since B^+ is the closure of $A_{\text{inf}}[1/p]$ in B, we deduce that this isomorphism agrees with the restriction of φ on B^+, thereby completing the proof. □

Remark. Let us give an alternative description of the ring $\widetilde{B^+}$ and its Frobenius automorphism. We define the Gauss 1-norm on $A_{\text{inf}}[1/p]$ by

$$\left|\sum [c_n]p^n\right|_1 := \sup_{n\in\mathbb{Z}}(|c_n|) \qquad \text{for all } c_n \in \mathcal{O}_F.$$

By construction we have $|f|_1 = \lim_{\rho\to 1} |f|_\rho$ for every $f \in A_{\text{inf}}[1/p]$, and consequently find that the Gauss 1-norm is indeed a multiplicative norm. It is then straightforward to verify that $\widetilde{B^+}$ is naturally isomorphic to the completion of $A_{\text{inf}}[1/p]$ with respect to the Gauss 1-norm. Hence we may obtain φ^+ as a unique continuous extension of φ_{inf} by Lemma 1.2.15.

However, we avoid using this description because working with the Gauss 1-norm is often subtle. The main issue is that the natural map $\mathcal{O}_F \longrightarrow A_{\text{inf}}[1/p]$ given by the Teichmüller lifts is not continuous with respect to the Gauss 1-norm. In fact, it is not hard to show

$$\lim_{c\to 0} |[1+c]-1|_1 = 1 \neq 0.$$

Definition 4.1.10. We refer to the map φ^+ constructed in Proposition 4.1.9 as the *Frobenius automorphism* of $\widetilde{B^+}$. We often abuse notation and write φ for φ^+ and the Frobenius endomorphism of B_{cris}.

PROPOSITION 4.1.11. The Frobenius endomorphism of B_{cris} is injective.

PROOF. Proposition 4.1.9 implies that φ is injective on B^+_{cris}, and in turn yields the desired assertion as we have $B_{\text{cris}} = B^+_{\text{cris}}[1/t]$ and $\varphi(t) = pt$ by Proposition 3.1.11 from Chapter III. □

Remark. Proposition 4.1.11 proves Theorem 3.1.13 from Chapter III.

PROPOSITION 4.1.12. We have identities

$$B^+ = \bigcap_{n\geq 0} \varphi^n(B^+_{\text{cris}}) \qquad \text{and} \qquad B^+[1/t] = \bigcap_{n\geq 0} \varphi^n(B_{\text{cris}}).$$

PROOF. By Proposition 4.1.6 and Proposition 4.1.9 we have

$$B^+_{1/p^{p^{n+1}}} = \varphi^n(B^+_{1/p^p}) \subseteq \varphi^n(B^+_{\text{cris}}) \subseteq \varphi^n(B^+_{1/p}) = B^+_{1/p^{p^n}} \qquad \text{for every } n \geq 0,$$

and consequently find

$$B^+ = \bigcap_{\rho\geq 0} B^+_\rho = \bigcap_{n\geq 0} B^+_{1/p^{p^n}} = \bigcap_{n\geq 0} \varphi^n(B^+_{\text{cris}}).$$

The second identity then follows as we have $B_{\text{cris}} = B^+_{\text{cris}}[1/t]$ and $\varphi(t) = pt$ by Proposition 3.1.11 from Chapter III. □

PROPOSITION 4.1.13. For every $n \in \mathbb{Z}$, we have

$$B^{\varphi=p^n} = (B^+)^{\varphi=p^n} = (B^+_{\text{cris}})^{\varphi=p^n}.$$

PROOF. The first identity is an immediate consequence of Proposition 3.1.11. The second identity follows from Proposition 4.1.12. □

COROLLARY 4.1.14. We have $X = \text{Proj}\left(\bigoplus_{n\geq 0} (B^+_{\text{cris}})^{\varphi=p^n}\right)$.

PROPOSITION 4.1.15. There exists a canonical isomorphism $B_e \cong B[1/t]^{\varphi=1}$.

PROOF. Proposition 4.1.12 and Proposition 4.1.13 together yield a natural identification

$$B[1/t]^{\varphi=1} \cong B^+[1/t]^{\varphi=1} = B^{\varphi=1}_{\text{cris}} = B_e$$

as desired. □

COROLLARY 4.1.16. The ring B_e is a principal ideal domain.

PROOF. By construction, the element t induces the closed point ∞ on X. Hence we have an identification $X\setminus\{\infty\} \cong \text{Spec}\,(B[1/t]^{\varphi=1})$, and consequently deduce the assertion by Theorem 2.4.8. □

Remark. Corollary 4.1.16 was first proved by Fontaine prior to the construction of the Fargues-Fontaine curve. Fontaine's proof was motivated by a result by Berger [**Ber08**] that B_e is a Bézout ring, and eventually inspired the first construction of the Fargues-Fontaine curve as we will soon describe in the subsequent subsection.

4.2. Essential image of the crystalline functor

In this subsection, we describe the essential image of the functor D_{cris} using vector bundles on the Fargues-Fontaine curve. Our discussion will be cursory, and will focus on explaining some key ideas for studying p-adic Galois representations via vector bundles on the Fargues-Fontaine curve. Throughout this subsection, let us write $U := X \setminus \{ \infty \}$.

PROPOSITION 4.2.1. Let M_e be a free B_e-module of finite rank, and let M_{dR}^+ be a B_{dR}^+-lattice in $M_{\mathrm{dR}} := M_e \otimes_{B_e} B_{\mathrm{dR}}$.

(1) There exists a unique vector bundle $\mathcal{V}$ on X with

$$H^0(U, \mathcal{V}) \cong M_e \qquad \text{and} \qquad \widehat{\mathcal{V}_\infty} \cong M_{\mathrm{dR}}^+$$

where $\widehat{\mathcal{V}_\infty}$ denotes the completed stalk of $\mathcal{V}$ at ∞.

(2) The vector bundle $\mathcal{V}$ gives rise to a natural exact sequence

$$0 \longrightarrow H^0(X, \mathcal{V}) \longrightarrow M_e \oplus M_{\mathrm{dR}}^+ \longrightarrow M_{\mathrm{dR}} \longrightarrow H^1(X, \mathcal{V}) \longrightarrow 0$$

where the middle arrow maps each (x, y) to $x - y$.

Remark. The first statement is in fact a standard application of the Beauville-Laszlo theorem as stated in [**BL95**] or [**Sta**, Tag 0BP2]. The second statement then follows as a variant of the Mayer-Vietoris long exact sequence.

Example 4.2.2. By Proposition 4.1.2 and Proposition 4.1.15 we have natural identifications

$$H^0(U, \mathcal{O}_X) \cong B_e \qquad \text{and} \qquad \widehat{\mathcal{O}_{X,\infty}} \cong B_{\mathrm{dR}}^+$$

where $\widehat{\mathcal{O}_{X,\infty}}$ denotes the completed local ring at ∞. Hence by Therem 3.2.9 and Proposition 4.2.1 we obtain a natural exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \oplus B_{\mathrm{dR}}^+ \longrightarrow B_{\mathrm{dR}} \longrightarrow 0,$$

which in turn yields the fundamental exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \longrightarrow 0$$

as described in Chapter III, Theorem 3.1.14.

Remark. In fact, the Fargues-Fontaine curve was originally constructed by gluing Spec (B_e) and Spec (B_{dR}^+) using the fundamental exact sequence, partially motivated by Colmez's theory of Banach-Colmez spaces as developed in [**Col02**].

Definition 4.2.3. Let N be a filtered isocrystal over K. Let us write $\mathrm{rk}(N)$ and $\deg(N)$ respectively for the rank and the degree of N as an isocrystal over K_0.

(1) We define the *degree* of the filtered vector space N_K, denoted by $\deg(N_K)$, to be the unique integer d with $\mathrm{Fil}^d(\det(N_K)) \neq 0$.

(2) We define the *degree* of N by

$$\deg^\bullet(N) := \deg(N) - \deg(N_K).$$

(3) If N is not zero, we define its *slope* by

$$\mu^\bullet(N) := \frac{\deg^\bullet(N)}{\mathrm{rk}(N)}.$$

Remark. It is straightforward to verify that MF_K^φ is a slope category as remarked after Theorem 3.3.22. Hence every $N \in \mathrm{MF}_K^\varphi$ admits a unique Harder-Narasimhan filtration.

Example 4.2.4. Let V be a crystalline Γ_K-representation. We wish to show that $D_{\text{cris}}(V)$ has degree 0. Proposition 3.2.14 from Chapter III implies that $\det(V)$ is a crystalline Γ_K-representation with $\det(D_{\text{cris}}(V)) \cong D_{\text{cris}}(\det(V))$, and consequently yield

$$\deg^\bullet(D_{\text{cris}}(V)) = \deg^\bullet(\det(D_{\text{cris}}(V))) = \deg^\bullet(D_{\text{cris}}(\det(V))).$$

Hence we may replace V with $\det(V)$ to assume $\dim_{\mathbb{Q}_p} V = 1$.

Let us choose a continuous character $\eta : \Gamma_K \longrightarrow \mathbb{Q}_p^\times$ with $V \simeq \mathbb{Q}_p(\eta)$. Proposition 2.4.4 and Proposition 3.2.8 from Chapter III together imply that V is Hodge-Tate with

$$D_{\text{cris}}(V)_K \cong D_{\text{dR}}(V) \qquad \text{and} \qquad \text{gr}(D_{\text{dR}}(V)) \cong D_{\text{HT}}(V).$$

Hence Proposition 1.1.13 from Chapter III yields an integer n such that $\eta\chi^n(I_K)$ is finite. It follows by Theorem 1.1.8 from Chapter III that n is the Hodge-Tate weight of V, which in turn implies $\deg(D_{\text{cris}}(V)_K) = n$.

It remains to show that $D_{\text{cris}}(V)$ has degree n as an isocrystal. Let us denote by K^{un} the maximal unramified extension of K in $\overline{K}$, and by $\widehat{K^{\text{un}}}$ the p-adic completion of K^{un}. We also write $W(\overline{k})$ for the ring of Witt vectors over $\overline{k}$, and $\widehat{K_0^{\text{un}}}$ for the fraction field of $W(\overline{k})$. Example 3.2.2 and Proposition 3.2.13 from Chapter III together imply that $V(n) \simeq \mathbb{Q}_p(\eta\chi^n)$ is crystalline with

$$D_{\text{cris}}(V(n)) \cong D_{\text{cris}}(V) \otimes_K D_{\text{cris}}(\mathbb{Q}_p(n)). \tag{4.6}$$

We then find by Example 3.2.9 from Chapter III that $\eta\chi^n(I_K)$ is trivial. Moreover, by construction $\widehat{K^{\text{un}}}$ is a p-adic field with I_K as the absolute Galois group. Therefore we have

$$D_{\text{cris}}(V(n)) = (V(n) \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_K} \subseteq (V(n) \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{I_K} \cong B_{\text{cris}}^{I_K} \cong \widehat{K_0^{\text{un}}}$$

where the last identification follows from Theorem 3.1.8 from Chapter III. It follows by Proposition 3.2.7 from Chapter III that the Frobenius automorphism of $D_{\text{cris}}(V(n))$ extends to the Frobenius automorphism of $\widehat{K_0^{\text{un}}}$, which in turn implies that $D_{\text{cris}}(V(n))$ has degree 0 as an isocrystal. In addition, as we have $\varphi(t) = pt$ by construction, we deduce by Example 3.2.2 from Chapter III that $D_{\text{cris}}(\mathbb{Q}_p(n))$ has degree $-n$ as an isocrystal. The assertion is now straightforward to verify by the natural isomorphism (4.6) in MF_K^φ.

Definition 4.2.5. Let N be a filtered isocrystal over K.

(1) We say that N is *semistable* if we have $\mu^\bullet(M) \leq \mu^\bullet(N)$ for every nonzero filtered subisocrystal M of N.
(2) We say that N is *weakly admissible* if it is semistable of slope 0.
(3) We say that N is *admissible* if it is in the essential image of D_{cris}.

Proposition 4.2.6. Every admissible filtered isocrystal over K is weakly admissible.

Remark. The proof of Proposition 4.2.6 is mostly an elementary algebra, after replacing K by the completion of the maximal unramified extension of K in light of the remark after Proposition 3.2.20 from Chapter III. Curious readers can find a detailed proof in [**BC**, Theorem 9.3.4].

Proposition 4.2.7. Let N be a weakly admissible filtered isocrystal over K, and set

$$V := (N \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \cap \text{Fil}^0(N_K \otimes_K B_{\text{dR}}).$$

(1) V is naturally a crystalline Γ_K-representation with $\dim_{\mathbb{Q}_p}(V) \leq \dim_{K_0}(N)$.
(2) N is admissible if and only if we have $\dim_{\mathbb{Q}_p}(V) = \dim_{K_0}(N)$.

Remark. We refer the readers to [**BC**, Proposition 9.3.9] for a complete proof. If N is admissible, the assertions are evident by Proposition 3.2.18 from Chapter III.

PROPOSITION 4.2.8. Let N be a filtered isocrystal over K.

(1) There exists a unique vector bundle $\mathcal{E}'(N)$ on X with
$$H^0(U, \mathcal{E}'(N)) \cong (N \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \quad \text{and} \quad \widehat{\mathcal{E}'(N)_\infty} \cong \text{Fil}^0(N_K \otimes_K B_{\text{dR}})$$
where $\widehat{\mathcal{E}'(N)_\infty}$ denotes the completed stalk of $\mathcal{E}'(N)$ at ∞.

(2) We have $\text{rk}(N) = \text{rk}(\mathcal{E}'(N))$, $\deg^\bullet(N) = \deg(\mathcal{E}'(N))$ and $\mu^\bullet(N) = \mu(\mathcal{E}'(N))$.

(3) N is weakly admissible if and only if $\mathcal{E}'(N)$ is semistable of slope 0.

Remark. A complete proof of Proposition 4.2.8 may be added later. Here we explain some key ideas as sketched in [**FF18**, Lemma 10.5.5 and Proposition 10.5.6].

The first statement follows from Proposition 4.2.1 once we verify verify using Theorem 2.3.24 from Chapter II that $(N \otimes_{K_0} B_{\text{cris}})^{\varphi=1}$ is a free B_e-module with an identification
$$(N \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \otimes_{B_e} B_{\text{dR}} \cong N_K \otimes_K B_{\text{dR}}.$$
The second statement can be obtained by realizing $\mathcal{E}'(N)$ in a short exact sequence
$$0 \longrightarrow \mathcal{E}'(N) \longrightarrow \mathcal{E}(N) \longrightarrow \mathcal{T} \longrightarrow 0$$
where $\mathcal{T}$ is a torsion sheaf supposed at ∞. The third statement is obtained as a special case of the fact that the functor $\mathcal{E}'$ preserves the Harder-Narasimhan filtration, which is not hard to prove by observing that the Harder-Narasimhan filtrations of N and $\mathcal{E}'(N)$ are stable under the natural actions of Γ_K.

THEOREM 4.2.9 (Colmez-Fontaine [**CF00**]). A filtered isocrystal N over K is admissible if and only if it is weakly admissible.

PROOF. If N is admissible, then it is weakly admissible by Proposition 4.2.6. Let us now assume that N is weakly admissible, and set
$$V := (N \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \cap \text{Fil}^0(N_K \otimes_K B_{\text{dR}}).$$
In light of Proposition 4.2.7, it suffices to show $\dim_{\mathbb{Q}_p}(V) = \dim_{K_0}(N)$. Proposition 4.2.8 yields a semistable vector bundle $\mathcal{E}'(N)$ on X of slope 0 with
$$H^0(U, \mathcal{E}'(N)) \cong (N \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \quad \text{and} \quad \widehat{\mathcal{E}'(N)_\infty} \cong \text{Fil}^0(N_K \otimes_K B_{\text{dR}})$$
where $\widehat{\mathcal{E}'(N)_\infty}$ denotes the completed stalk of $\mathcal{E}'(N)$ at ∞. Hence by Proposition 4.2.1 we obtain a canonical isomorphism
$$H^0(X, \mathcal{E}'(N)) \cong (N \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \cap \text{Fil}^0(N_K \otimes_K B_{\text{dR}}) = V.$$
Moreover, Theorem 3.5.3 and Proposition 4.2.8 together imply that $\mathcal{E}'(N)$ is isomorphic to $\mathcal{O}_X^{\oplus r}$ where we set $r := \dim_{K_0}(N)$, and consequently yields an isomorphism
$$V \cong H^0(X, \mathcal{E}'(N)) \simeq H^0(X, \mathcal{O}_X)^{\oplus r} \cong \mathbb{Q}_p^{\oplus r}$$
by Proposition 3.1.6 and Theorem 3.2.9. We thus find $\dim_{\mathbb{Q}_p}(V) = \dim_{K_0}(N)$ as desired. □

Remark. While the proof above greatly simplifies the original proof by Colmez-Fontaine [**CF00**] and another proof by Berger [**Ber08**], these prior proofs contained a number of important ideas that contributed to the discovery of the Fargues-Fontaine curve.

COROLLARY 4.2.10. The functor D_{cris} is an equivalence between $\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$ and the category of weakly admissible filtered isocrystals over K.

PROOF. This is immediate by Theorem 3.2.19 from Chapter III and Theorem 4.2.9. □

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